CHE656

Process Analysis and Modeling

Modeling and Optimizing Chemical Processes

Exercise Problems

Year 2025 Edition

Prepared by

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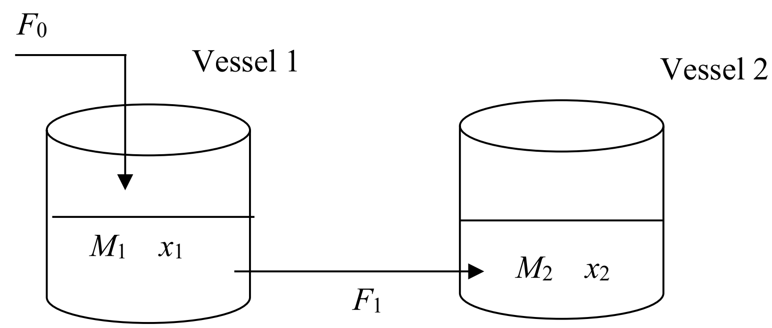
Part I

Process Modeling and Dynamics

1. ***Mass Balance in a Two-Vessel Flow Tank System, Part I***

Consider two vessels connected in series as shown in the figure below in which M_1, x_1 and M_2, x_2 represent the total mass (in kg) and the mass fraction of methanol in Vessel 1 and Vessel 2, respectively. Vessel 1 is initially filled with 50 kg of pure methanol, while Vessel 2 is filled with 10 kg of pure water. A mixture of methanol-water solution (with 40 mass% methanol) flows into Vessel 1 at a mass flow rate F_0 = 1.0 kg/min. The solution in Vessel 1 then flows into Vessel 2 with a mass flow rate F_1 = 2.0 kg/min.



(a) Derive analytically an expression of x_1, the mass fraction of methanol in Vessel 1, as a function of time.

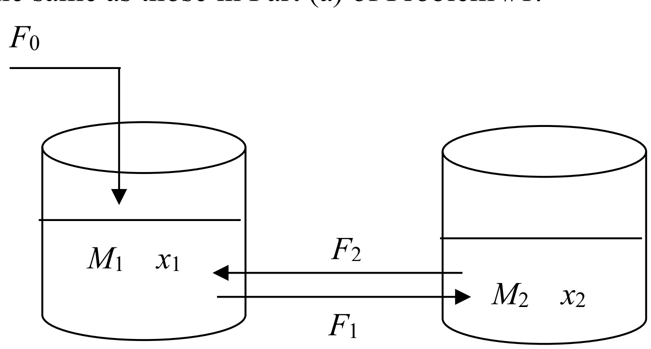
(b) Derive analytically an expression of x_2, the mass fraction of methanol in Vessel 2, as a function of time. Note that you must do the integration by hand by showing your work. Do not obtain a one-step solution from the Internet.

(c) It is clear that the concentration of methanol x_2 will go through a maximum because the concentration of methanol in Vessel 1 that flows into Vessel 2 becomes more dilute over time. Determine the time at which this maximum occurs and the maximum value of x_2.

Answers: t_{max} = ____________ minutes; $x_{2,max}$ = ____________

2. ***Mass Balance in a Two-Vessel Flow Tank System, Part II***

(a) Now consider the two-vessel in Problem #1 with another scenario in which there is a recycle as shown. The recycle flow rate F_2 = 1.0 kg/min, while F_0 and F_1 remain the same at 1.0 kg/min and 2.0 kg/min, respectively. The initial conditions in the two vessels are the same as those in Part (a) of Problem #1.



Derive a single ODE expressing x_2 as a function of time using mass balance around the two vessels. Note that this ODE will be 2nd-order.

(b) Use *ode45* in MATLAB to solve the ODE in Part (a) going from $t = 0$ to $t = 30$ minutes. Also, confirm your answers by solving the two derived 1st-order ODEs simultaneously. The two sets of answers should come out to be the same.

Answers: x_2 (t = 1 min) = ____________; x_2 (t = 10 min) = ____________

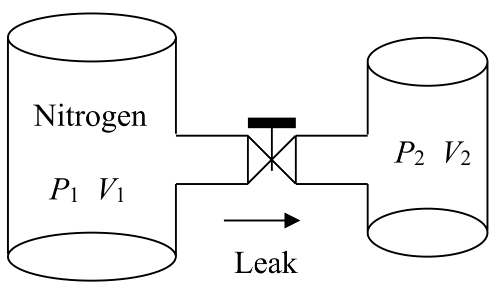
3. *A Two-Tank Gas System*

A large tank is connected to a smaller tank by means of a valve, which remains closed at all times. The large tank contains nitrogen (assume it is an ideal gas) at 700 kPa, while the small tank is evacuated (vacuum). The valve between the two tanks starts to leak and the rate of gas leakage is proportional to the pressure difference between the two tanks, as given by

$$Leak\ \ Flow = C_V \sqrt{P_{AVG}(P_1 - P_2)}$$

where P_{AVG} = average absolute pressure across the valve = $(P_1 + P_2)/2$

Valve constant $C_V = 3.6\times10^{-4}$ kmol/kPa-hour.
Tank volumes are 30 m^3 and 15 m^3.
Assume constant temperature of 293.15 K in both tanks.
Universal gas constant = 8.31439 kPa-m^3/kmol-K.



(a) Derive an analytical expression of dn_1/dt as a function of n_1, where n_1 is the number of moles of nitrogen in the large tank. This ODE of yours should contain only n_1 as the unknown variable. Use MATLAB to solve for n_1 (use *ode45* and "format short") and run the model up until $t = 25$ hours.

Answer: n_1 (t = 25 hours) = _______________ kmoles

(b) As a good chemical engineer, you always look for ways to simplify your engineering calculations. In this problem, it is possible to derive an analytical solution when t is small, i.e. when $P_2 << P_1$. Derive an analytical expression for n_1 as a function of time t.

Calculate n_1 after 2 hours and 4 hours and compare these solutions with those from Part (a) by reporting the absolute % relative errors (assuming the MATLAB solutions are exact and show 4 decimal places).

Answers:

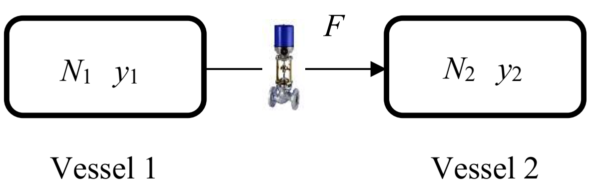
n_1 (t = 2 hours) = ____________ kmoles   % relative error = __________

n_1 (t = 4 hours) = ____________ kmoles   % relative error = __________

4. *Mass Balance in a Two-Vessel System*

Consider the following two closed vessels connected in series used to store gas. Initially (at $t = 0$), Vessel 1 is filled with 100 gmoles of a mixture of methane and nitrogen, 80 mol% of which is methane, while Vessel 2 is filled with 10 gmoles of pure nitrogen.

The gas in Vessel is gradually released into Vessel 2 via a sophisticated control valve which allows the released molar flow rate F (gmoles/minute) to be directly proportional to the amount of gas left in Vessel 1, i.e $F = 0.01N_1$, where N_i is the total number of gmoles of gas in Vessel i and y_i is the mole fraction of methane in Vessel i.



(a) Derive an analytical expression of N_2, the total amount of gas in Vessel 2, as a function of time.

Answer: N_2 (t = 30 min) = ___________ gmoles

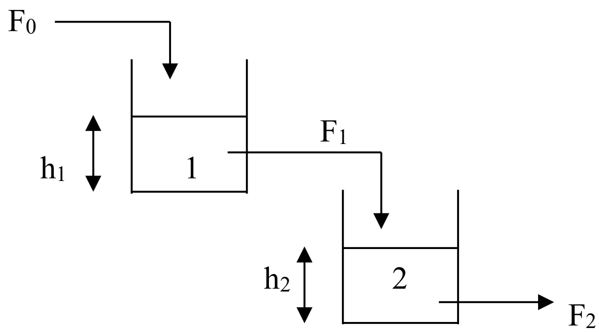
(b) Derive an analytical expression of y_2, the mole fraction of methane in Vessel 2, as a function time. You must show detailed work in your integration and are not allowed to use formulas from the Internet or tables of integration.

Answer: y_2 (t = 30 min) = ___________

(c) Verify your analytical solution in Part (b) by solving the ODE dy_2/dt derived using *ode45* solver in MATLAB. Make a plot of x_2 as a function of time from $t = 0$ to $t = 200$ minutes. Hint: Another way to verify your MATLAB solution is to specify an initial condition of $y_2(0) = 0.8$, which will result in a flat response because y_2 will no longer vary with time.

5. ***Two Open Continuous Flow Tanks in Series***

Consider two tanks in series as shown where the flow out of the first tank enters the second tank. Our objective is to develop a model to describe how the height of liquid in tank 2 changes with time, given the input flowrate $F_0(t)$. Assume that the flow out of each tank is a linear function of the height of liquid in the tank (i.e. $F_1 = \beta_1 h_1$ and $F_2 = \beta_2 h_2$) and each tank has a constant cross-sectional area.



(a) Develop a system of equations to describe the liquid height of the two tanks as a function of time, assuming constant liquid density. Be sure to define all your variables.

(b) Solve for h_1 and h_2 analytically if

$A_1 = 10$ ft^2 $\quad$ $A_2 = 20$ ft^2 $\quad$ $F_0 = 2$ ft^3/hr

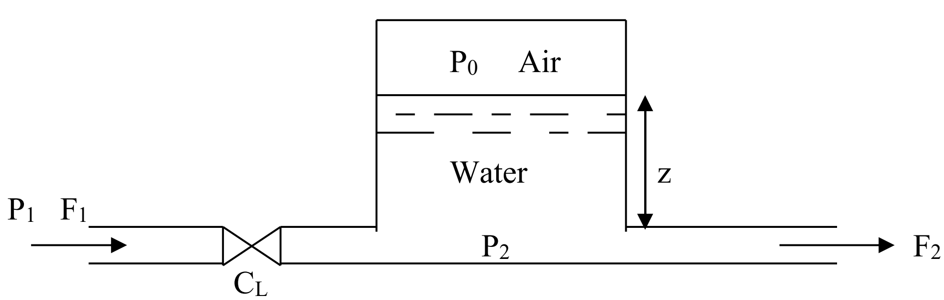
$\beta_1 = \beta_2 = 1$ ft^2/hr $\quad$ where A_i is the cross-sectional area of tank i

and tank 1 initially has a liquid height of 1 ft of while tank 2 is empty. Plot the two liquid height profiles as a function of time from 0 hr to 5 hr.

(c) Suppose F_0 is no longer constant but equals to $2t + 1$ ft^3/hr. Use MATLAB to solve for the liquid heights in both tanks. Plot both profiles as a function time up to 5 hr.

6. ***Adiabatic Expansion/Compression in an Enclosed Vessel***

Consider the following system in which air is trapped inside an enclosed vessel while water flows into the vessel at a rate of F_1 and flows out at a rate of F_2.



Initially, the air is at 25°C and 1.01325×10^5 Pa (Pascal), and the liquid height is 5.0 meter. The following data are available:

Gravitational acceleration $g = 9.80665$ m/s^2
Air C_V (heat capacity at constant volume) = 2.0731×10^4 J/kmol°C

Vessel volume =10.0 m^3	Vessel cross-sectional area = 1.0 m^2
Water density = 1000 kg/m^3	Gas constant R = 8314 m^{3}Pa/kmol°K
	= 8314 J/kmol°K
Air MW = 29.0 kg/kmol	Valve constant C_L = 0.001 m^3/Pa$^{1/2}$min
Outlet flow rate F_2 = 0.1 m^3/min	Source pressure $P_1 = 3.0\times10^5$ Pa

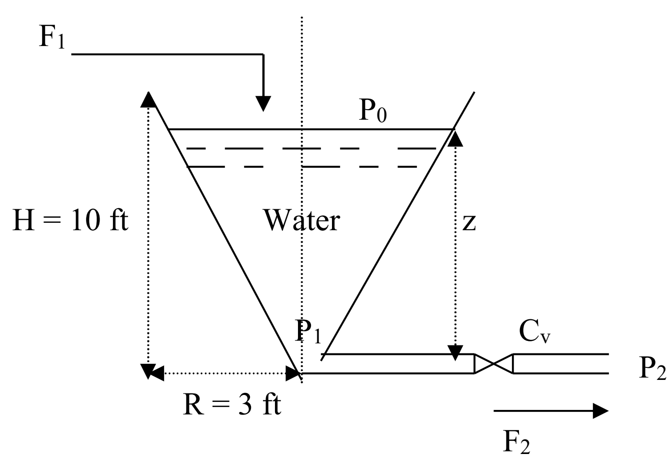
(a) Assuming adiabatic expansion/compression due to the rise/fall of the liquid height, write down all the system-describing equations. Be sure to define all your variables which are unknown. Use the data given and simplify each equation as much as possible.

(b) Derive an ordinary differential equation that relates the liquid height as a function of time. Your final equation must contain Z as the only dependent variable.

(c) Solve the ODE in Part (b) using MATLAB and plot the liquid height z as a function of time.

7. *Continuous Cone-Shaped Open Vessel*

Consider a conical open vessel as shown in the following figure. The vessel has a radius R at the top and a height H. Water flows into the vessel at a rate of F_1 and it flows out through a valve at a rate of F_2. The following data are given:



$P_0 = P_2$ = 14.7 psia
C_v = characteristic valve constant = 3.0 ft^3/psia$^{1/2}$-min
ϕ = water mass density = 62.4 lbm/ft^3
z_0 = initial liquid height = $z(t = 0)$ = 5 ft

F_1 is not constant but is controlled such that it varies with the liquid height according to:

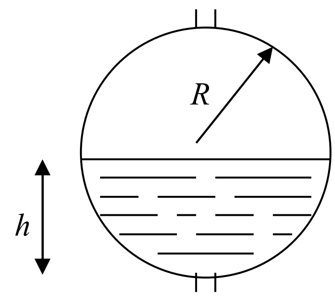
$$F_1 = 2z^{1/2} \quad ft^3/min$$

Solve for z analytically as a function of time.

Hint: Volume of a cone $V = \int_0^z A(z)dz$
where A(z) = cone cross-sectional area and is a function of z

8. *Liquid Height in a Spherical Tank*

The following figure shows a spherical tank for storing water. The tank is filled through a hole in the top and drained through a hole in the bottom. If the tank's radius is *R*, one can



use integration to show that the volume of water in the tank as a function of its height *h* is given by:

$$V(h) = \pi R h^2 - \frac{\pi h^3}{3}$$

Studies in fluid mechanics have identified the relation between the volume flow through the bottom hole and the liquid height as:

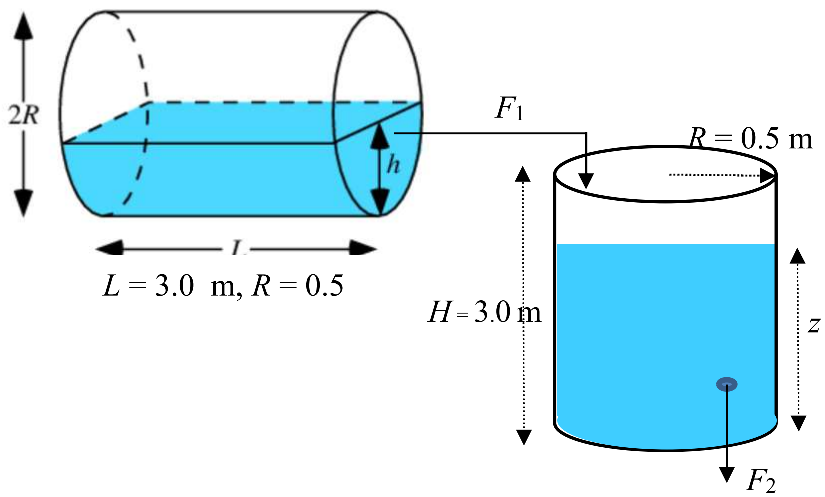
$$F = C_d A\sqrt{2gh}$$

where *A* is the area of the hole, *g* is the acceleration due to gravity (32.2 ft/sec^2), and C_d is an experimentally determined value that depends partly on the type of fluid (for water, C_d = 0.6).

Suppose the initial height of water is 8 ft and the tank has a radius of 5 ft with a 1-inch diameter hole in the bottom. Use MATLAB to determine how long it will take for the tank to empty (i.e. drain completely), to the nearest second (i.e. no decimal places).

9. *Dynamics of Liquid Height in Two Vessels in Series*

A horizontal cylindrical vessel and a vertical cylindrical vessel with the given dimensions are connected in series as shown.



Water flows out of the horizontal cylindrical vessel with F_1 = 1.0 m^3/min which then flows into the second vessel. There is a small spherical hole (radius = 2 cm or 0.02 m) at the bottom in the second vessel which drains the water based on gravity at a volumetric flow rate of:

$$F_2 = C_d A\sqrt{2gz}$$

where A is the area of the hole, g (9.81 m/s^2 or 35316 m/min^2) is the acceleration due to gravity, and C_d is an experimentally determined value that depends partly on the type of fluid (for water, C_d = 0.6).

The following very complicated formula has been derived for the volume of a horizontal cylindrical vessel which is partially filled as a function of L, R and h.

$$V(L,R,h) = L\left[R^2\cos^{-1}\left(\frac{R-h}{R}\right) - (R-h)\sqrt{2Rh-h^2}\right]$$

(a) At t = 0, the horizontal cylindrical vessel is three-quarter full. Calculate the height above the bottom of the horizontal cylindrical vessel at t = 0.

h when the horizontal cylindrical vessel is three-quarter full = _____________ meter

(b) Derive an ODE that expresses h as a function of time in the horizontal cylindrical vessel. You must simplify the ODE as much as possible. Solve the ODE by *ode45* and plot the dynamics of the liquid height h and determine the time it takes for the vessel to become completely empty. You will need the following formula:

$$\frac{d(\cos^{-1}x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

t when the cylindrical vessel is empty = _______________ minutes

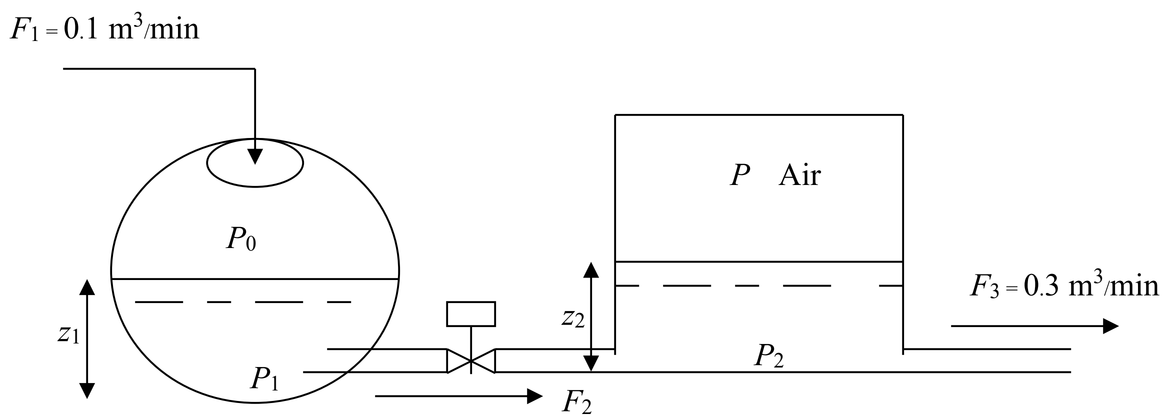
(<u>Hint</u>: You could use simple volume calculations to verify your MATLAB answer.)

(c) Derive an analytical solution for the liquid height z in the vertical cylindrical vessel as a function of time, given that at $t = 0$, the vertical vessel is one-half full.

z at $t = 1$ minute = _______________ meters

10. *Modeling Liquid Heights in a Two-Vessel System*

An open spherical vessel is connected to a closed rectangular tank through an open valve as shown in the figure below. Water flows continuously into the spherical vessel at a volumetric flow rate of F_1 with some of the water in the sphere also flowing into the rectangular tank at a rate of F_2. Water then flows out of the rectangular tank at a flow rate of F_3, and the temperature inside the tank is always maintained at 298.15 K. We wish to study the dynamics of the liquid heights in the two vessels using MATLAB.



The following data are known about the system:

$C_V = 1\times10^{-5}$ m^3/Pa$^{1/2}$-min (characteristic valve constant)
Cross-sectional area of rectangular tank = 3 m^2
Height of rectangular tank = 10 m
Initially at $t = 0$, $z_1 = 3$ m, $z_2 = 2$ m, $P = 1.01325\times10^5$ Pascal
Gravitational acceleration $g = 9.80665$ m/s^2

$P_0 = 1.01325\times10^5$ Pascal
Radius of sphere $R = 2$ m
ϕ (water) = 1000 kg/m^3

Universal gas constant $R = 8314$ m^3-Pa/kmol-K = 8314 J/kmol-K
Liquid volume inside a sphere as a function of liquid height z is given by:

$$V = \pi R z^2 - \frac{\pi z^3}{3}$$

Use MATLAB's *ode*45 to determine whether the liquid heights in the two vessels will ever be equal, i.e. $z_1 = z_2$. If so, report the time (accurate to one decimal place) at which this happens and the height. If z_1 is never equal to z_2, determine when the two liquid heights are at the closest. Run your model for 20 minutes, which should be sufficient to answer the questions.

Answer the following questions:

Are the two liquid heights ever equal? ☐ Yes ☐ No

If yes, $z_1 = z_2 =$ ____________ meters

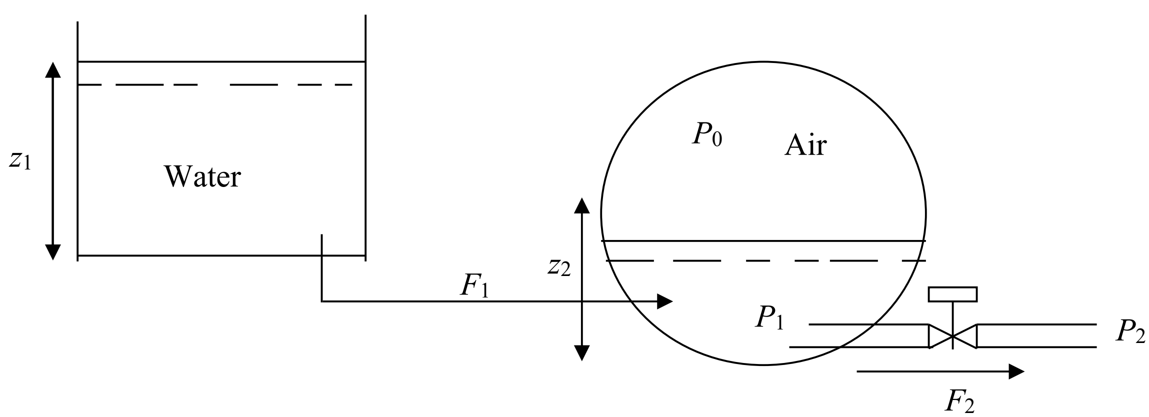
Time at which the two liquid heights are equal or at the closest = _____________ minutes

11. *Modeling Liquid Heights in a Two-Vessel System*

An open rectangular vessel is connected to a well-insulated (i.e. adiabatic) and closed spherical vessel shown in the figure below. Water from the rectangular vessel is drained through a hole (radius = 4 cm) at its bottom into the spherical vessel at a volumetric flow rate of:

$$F_1 = C_d A \sqrt{2 g z_1}$$

where A is the area of the hole, g is the acceleration due to gravity, and C_d is an experimentally determined value that depends partly on the type of fluid (for water, C_d = 0.6). Water then flows out of the spherical vessel at a flow rate of F_2. We wish to study the dynamics of the liquid heights in the two vessels using MATLAB.



The following data are known about the system:

$C_{valve} = 1\times10^{-3}$ m^3/Pa$^{1/2}$-min (characteristic valve constant)
Cross-sectional area of the rectangular vessel = 4 m^2
Radius of sphere R = 2 m $P_2 = 1.01325\times10^5$ Pascal ϕ (water) = 1000 kg/m^3
C_V (air heat capacity at constant volume) = 20850 J/kmol-K

Initially at t = 0, z_1 = 6 m, z_2 = 2 m, $P_0 = 1.01325\times10^5$ Pascal, T_G = 303.15 K
Gravitational acceleration g = 9.80665 m/s^2 = 35303.94 m/min^2

Universal gas constant R = 8314 m^3-Pa/kmol-K = 8314 J/kmol-K
Conversion factors: 1 Pascal = 1 N/m^2 = 1 kg/m-s^2, 1 N = kg-m/s^2, 1 J = kg-m^2/s^2
Liquid volume inside a sphere as a function of liquid height z is given by:

$$V = \pi \ R \ z^2 - \frac{\pi \ z^3}{3}$$

Derive an ODE for z_2 as a function of time (without z_1 in the ODE) and use MATLAB's *ode*45 to simulate its dynamics until either the rectangular vessel dries up or the spherical vessel is full. You must simplify your final ODE as much as possible before using it in MATLAB. Note that *ode45* may result in solutions with imaginary parts. You can ignore them as long as the values are small which come from inherent problems in numerical integrations. **Hint:** Be very careful with your units and their conversions.

Answer the following questions:

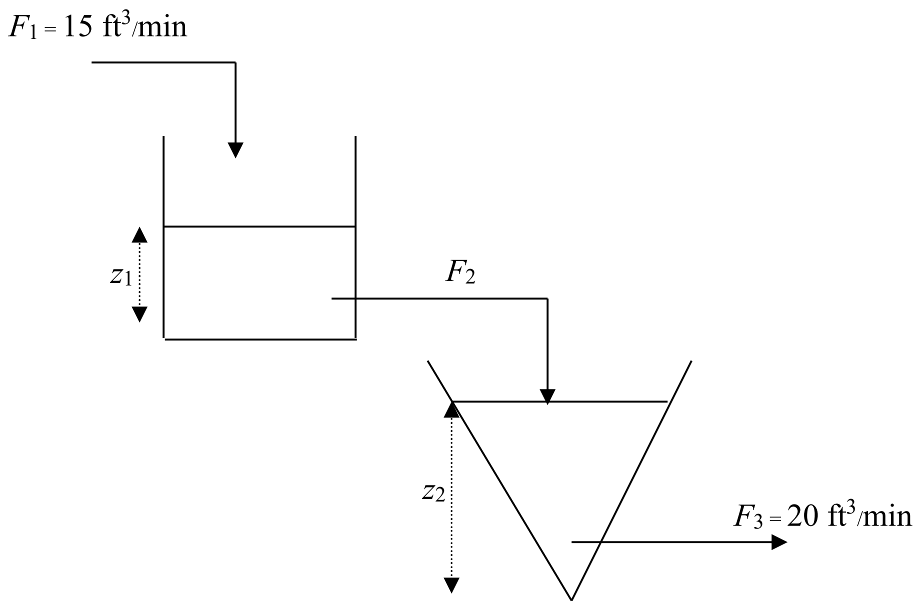
Time at which the rectangular vessel dries up or the spherical vessel is full (whichever is smaller) = ____________ minutes

12. *Modeling Liquid Heights in a Two-Vessel System*

Consider two tanks in series as shown where the water flow out of the first tank enters the second tank. The first tank is a cubic vessel with a width of 10 ft, a length of 10 ft, and a height of 10 ft, whereas the second tank is a cone-shaped vessel with a radius of 5 ft at the top and a height of 20 ft. The first tank is filled with water at a volumetric flow rate F_1 and is drained through a hole (radius = 1 inch) at the bottom. Studies in fluid mechanics have identified the relation between the volume flow through the bottom hole and the liquid height as:

$$F_2 = C_d A \sqrt{2 g z_1}$$

where A is the area of the hole, g is the acceleration due to gravity (32.2 ft/sec^2), and C_d is an experimentally determined value that depends partly on the type of fluid (for water, C_d = 0.6). Initially at t = 0, the cubic tank is filled with 2 ft of water and the cone-shaped tank is filled with 15 ft of water.



(a) Derive an analytical expression of liquid height z_1 of the cubic tank as a function of time. Does z_1 ever reach the steady-state, and if so, what is this value? Based on your analytical answer, also comment on whether z_1 reaches a maximum, reaches a minimum, overflows, or goes to zero and the time for that to happen. In your derivation, you are not allowed to use tables of integrals to perform the integration. Instead, use substitution and be careful with your unit conversions.

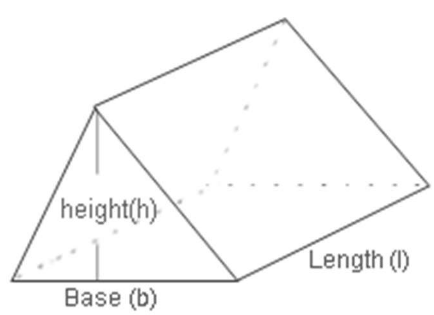
(b) Derive an ODE that describes the liquid height z_2 in the second tank. Together with the ODE for z_1 in Part (a), use MATLAB (*ode*45) to solve for z_2 as a function of time and plot both z_1 and z_2 as a function of time. Run the simulation for 20 minutes.

Answer the following questions:

z_1 (t = 10 minutes) = ____________ ft $\quad$ z_2 (t = 10 minutes) = ____________ ft

13. *Dynamics of the Liquid Height in a Horizontal Triangular Prism Vessel*

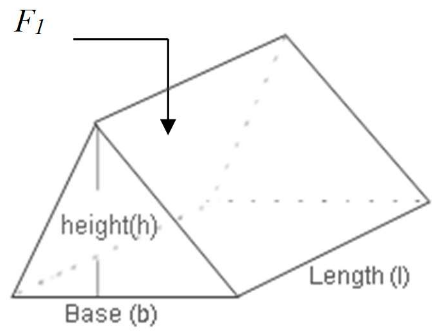
An open-topped (i.e. the top is open to the atmosphere) horizontal triangular prism vessel shown in the diagram below consists of the following dimensions: b, h, and l.



(a) Prove mathematically using simple geometry to show that when the vessel is filled with a liquid at height z, the total volume of the liquid inside the vessel is given by:

$$V = \frac{1}{2} bl \left[h - \frac{1}{h}(h - z)^2 \right]$$

(b) A continuous stream of water flows into the vessel at a rate of F_1 = 1 ft^3/min. Given that at t = 0, the vessel is empty and has the following dimensions, derive an analytical expression of z (explicitly) as a function of time (i.e. derive $z(t)$).



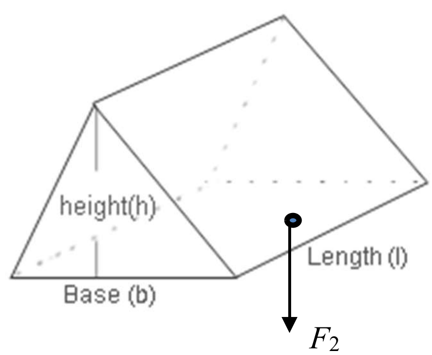
b = 0.5 ft, h = 2.0 ft, l = 20 ft

Answers: z (t = 5.0 mins) =__________ ft

t when the vessel is completely filled with water =__________ minutes

(c) Now, consider another scenario in which the vessel is fully filled with water at z = 2.0 ft when t = 0 with no inlet flow, and the water drains through a small hole at the bottom of the vessel due to gravity at a volumetric flow rate of (see the figure below):

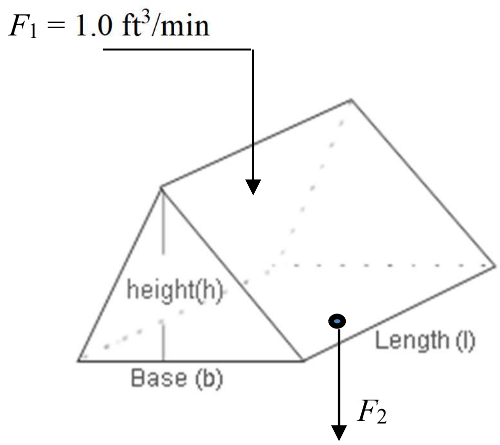
$$F_2 = C_d A_H \sqrt{2gz}$$



where A_H is the area of the hole with a radius of 1 inch or 0.0833 ft, g (32.174 ft/s^2 or 115826.58 ft/min^2) is the acceleration due to gravity, and C_d is an experimentally determined value that depends partly on the type of fluid (for water, C_d = 0.6). Derive an analytical expression of z as a function of time (note that this function will be implicit).

Answers: z (t = 1.0 minute) = __________ft; t (when z = 0) = _________ minutes

(d) Finally, consider the last scenario in which we combine Part (b) and Part (c) as shown in the figure below. The vessel is filled with water at z = 1.0 ft when t = 0.



Derive the following implicit but exact solution of z as a function of t for this scenario (the math is complicated but not impossible). This exact solution can be expressed as:

$$0.2388\left(1.2590\sqrt{z}-0.2\right)^2+0.2654\left(1.2590\sqrt{z}-0.2\right)^3 - 0.4983\ln\left(1.2590\sqrt{z}-0.2\right)-2.4283\left(1.2590\sqrt{z}-0.2\right)+2.0168 = t$$

Derive an ODE dz/dt and solve it in MATLAB using *ode45*. Compare the numerical answer from *ode45* with the answer from the analytical expression using *fsolve* at t = 1.0 minute.

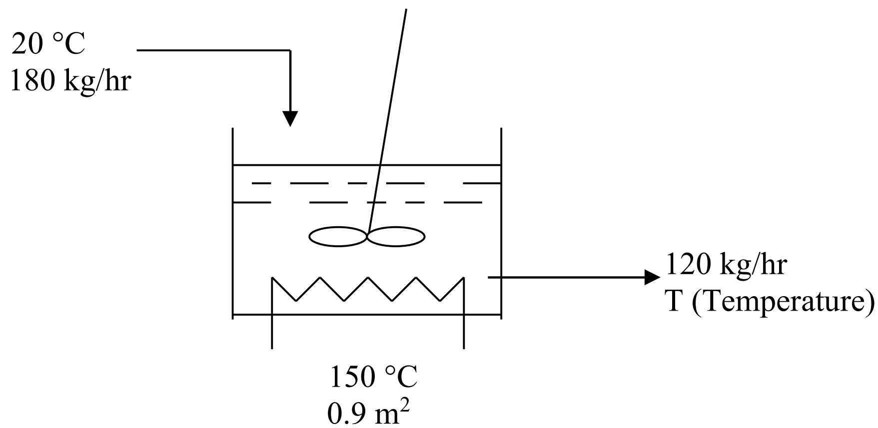
Answers: (4 decimal places)

z (t = 1.0 min, MATLAB) = __________ ft; z (t = 1.0 min, Analytical) = __________ ft

z (steady-state) = __________ ft

14. *Simultaneous Mass and Energy Balance*

Consider the following heating tank problem. A dilute solution at 20 °C is added to a well-stirred tank at the rate of 180 kg/hr. A heating coil having an area of 0.9 m^2 is located in the tank and contains steam condensing at 150 °C. The heated liquid leaves at 120 kg/hr and at the temperature of the solution in the tank. There is 500 kg of solution at 40 °C in the tank at the start of the operation. The overall heat-transfer coefficient is 342 kcal/hr-m^2-°C and the heat capacity of water is 1 kcal/kg-°C.



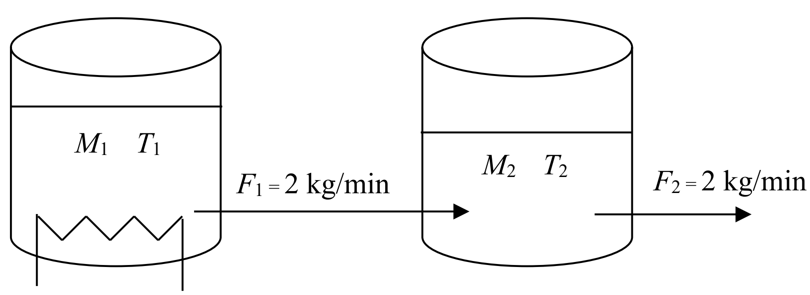
(a) Develop a system of mathematical equations to model this system. We are interested in determining the temperature of the solution in the tank at any given time.

(b) Solve for the temperature in the tank after 1 hour of heating. An analytical solution is possible for this particular system.

(c) Use MATLAB to solve for the temperature in the tank after 1 hour and compare the answer with the exact solution in Part (b).

15. *Mass and Energy Balance of Two Vessels Connected in Series*

Consider two vessels connected in series, the first of which is heated as shown. Both vessels are initially filled with water, but some water in the first vessel flows into the second vessel at the rate of F_1 while some water leaves the second vessel at the rate of F_2. The first vessel is heated by steam inside a heating coil which is always submerged at the bottom of the first vessel (so the heat transfer area can be assumed to be constant).



The following symbols are defined for all parameters and variables in this system. Data are also given for all symbols considered as parameters.

M_1 = Mass of water inside the first vessel (in kg) = 200 kg at $t = 0$
M_2 = Mass of water inside the second vessel (in kg) = 100 kg at $t = 0$

T_1 = Temperature of water inside the first vessel (in °C) = 20 °C at $t = 0$
T_2 = Temperature of water inside the second vessel (in °C) = 40 °C at $t = 0$
F_1 = Outflow rate from the first vessel to the second vessel = 2.0 kg/min
F_2 = Outflow rate from the second vessel = 2.0 kg/min
T_S = Steam temperature in the first vessel = 100 °C
U = Overall heat transfer coefficient of the heating coil = 4.0 kcal/m^2-min-°C
A = Heat transfer area of the heating coil (in m^2) = 1.0 m^2
C_P = Heat capacity of water = 1.0 kcal/kg-°C
ϕ = Mass density of water = 1000 kg/m^3

(a) Derive an analytical expression of T_1 as a function of time. Answer the following question:

T_1 after 30 minutes = ___________ °C

(b) Derive an explicit analytical expression of T_2 as a function of time. Simplify your final equation as much as possible, and answer the following questions:

T_2 after 30 minutes = ___________ °C

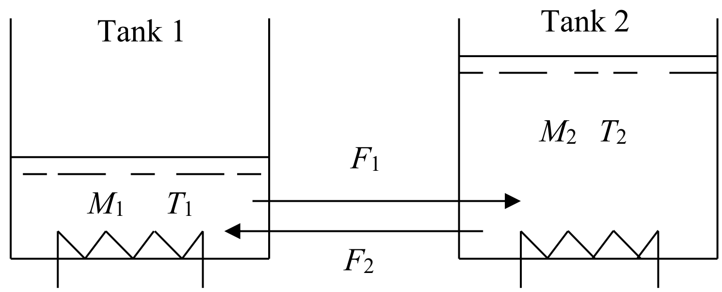
T_2 when the first vessel becomes empty = ____________ °C

(c) Solve for T_1 and T_2 again, this time numerically using *ode45* in MATLAB, if the first vessel has a radius of 0.5 m and is heated via a jacketed steam chamber (T_S = 100 °C). The heat transfer area consists of the bottom and the surface area around the vessel. As a result, this heat transfer area cannot be treated as constant. Answer the following questions:

T_1 after 30 minutes = ____________ °C
T_2 after 30 minutes = ____________ °C

16. *Energy Balance in a Two-Tank System*

Consider the following two-tank system in which water in the first tank flows into the second one with a flow rate of F_1 = 2 kg/min and vice versa with a flow rate F_2 = 2 kg/min. Water in both vessels is being heated with a heating coil with the same constant heat transfer area of A = 0.5 m^2 and the same heat transfer coefficient of U = 4 kcal/m^2-min-°C. However, the temperature of the heating coil in Tank 1 (100 °C) is lower than that of the heating coil in Tank 2 (120 °C). We define M_1, T_1, M_2, and T_2 as the mass and the temperature of water in the first vessel and in the second vessel, respectively.



(a) Using the following data about the system, derive two ODEs that describe T_1 and T_2 as a function of time.

C_P (water) = 1.0 kcal/kg-°C

Initially at $t = 0$: $M_1 = 50$ kg of water, $T_1 = 20$ °C, $M_2 = 100$ kg of water, $T_2 = 10$°C

(b) Solve the two ODEs in Part (a) and derive an analytical expression for T_1 and T_2 as a function of time.

(c) What is the domain of this system, i.e. the maximum time the derived model is valid for? Also, determine the time at which the temperatures in Tank 1 and Tank 2 are equal.

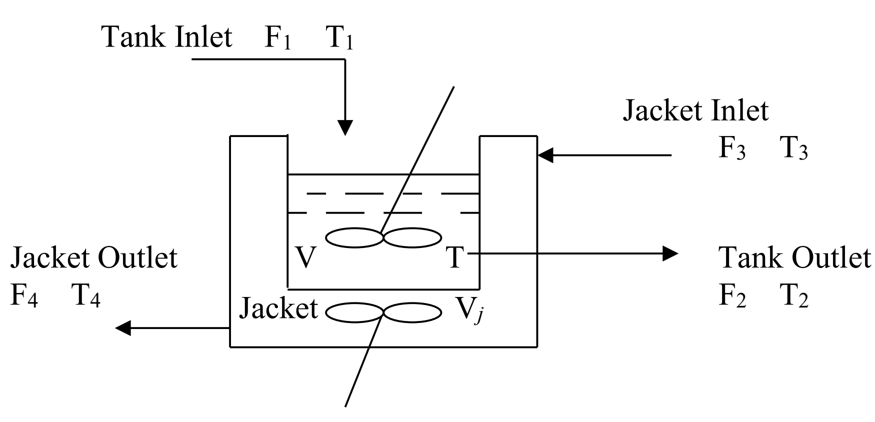
<u>Answer the following questions</u>:

Domain of this system = ____________ minutes

$T_1 = T_2 =$ ____________ °C when $t =$ ____________ minutes

17. *Mass and Energy Balance in a Stirred Tank Heater*

Consider the following stirred tank heater shown below, where the tank inlet stream is received from another process unit. A heat transfer fluid is circulated through a jacket to heat the fluid in the tank. Assume that no change of phase occurs in either the tank liquid or the jacket liquid. The following symbols are used: F_i = volumetric flowrate of stream i, and T_i = temperature of stream i.



Additional assumptions are:

1. The liquid levels in both the tank and the jacket are constant.
2. There is perfect mixing in both the tank and the jacket.
3. The rate of heat transfer from the jacket to the tank is governed by the equation $Q = UA(T_4 - T_2)$, where U is the overall heat transfer coefficient and A is the area of heat exchange.

(a) Write the dynamic modeling equations (ODEs) to find the tank and jacket temperatures. Do not use any numerical values – leave these equations in terms of the process parameters and variables. Be sure to define any new symbols you introduce into the equations.

(b) Assume that both the tank fluid and the jacket fluid are water. The steady-state values of this system variables and some parameters are:

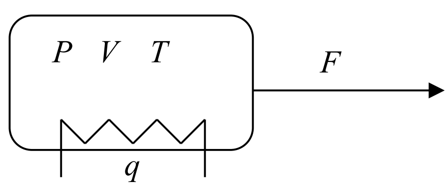
$F_1 = 1.0$ ft^3/min ρC_P (in tank) = ρC_P (in jacket) = 61.3 Btu/°F-ft^3
$T_1 = 50$ °F $T_2 = 125$ °F $V = 10$ ft^3
$T_3 = 200$ °F $T_4 = 150$ °F $V_j = 1$ ft^3

Solve for F_3 and UA (show units) at steady state.

(c) If initially (t = 0), $T_2 = 50$ °F and $T_4 = 200$ °F, solve for T_2 and T_4 from the ODEs in Part (a) analytically as a function of time.

18. *Mass and Energy Balance in a Gas Surge Drum*

Consider a heated gas storage drum in which its gas is being drawn out at the rate of *F* (in kmol/min). Assume ideal gas behavior in the drum and that heat is being added to the tank at the rate of *q*.



(a) Derive the modeling equations (ODEs) that describe how the temperature *T* and pressure *P* inside the drum vary with time. Note that for a gas, the accumulation term on the left-hand side of the energy balance equation is

$$\frac{dH}{dt} - \frac{d(PV)}{dt} = energy\,in \; - \; energy\,out$$

where $\frac{dH}{dt} = \frac{d(\rho C_P VT)}{dt}$, ρ is the molar density of the gas, and C_P is assumed constant.

For liquids, the $d(PV)/dt$ term is considered negligible (incompressible fluid and constant volume). But this is not true for gas so we cannot ignore the *PV* derivative term in the energy balance. Your two ODEs must be in terms of the following symbols: *P*, *V*, *T*, *R*, C_p, *F*, and *q*. Hint: to obtain the right ODEs, you must be able to identify which symbols are constants and which are variables.

(b) Calculate the time at which the drum will become completely empty. Then solve the two ODEs in the Part (a) using MATLAB, running the model for 20 minutes (soon after which the drum will become empty). The data are:

$V = 100$ m^3

C_P (gas) = 125 kJ/kmol-K

$q = 1.5\text{x}10^4$ kJ/min

$P(t = 0) = 1$ atm

$R = 0.08205$ m^3-atm/kmol-K
= 8.315 kJ/kmol-K

$F = 0.2$ kmol/min

$T(t = 0) = 298.15$ K = 25 °C

Notice that the gas constant *R* is given in two different units. Both numbers must be used, depending on where in the ODEs which requires some dimensional analysis.

Answer the following questions:

While it is obvious that the temperature profile in the drum will rise continuously, the pressure profile may go through a maximum, depending on the draw-out rate. Of course, *P* inside the drum will eventually drop to zero, but it may rise for a brief moment because of the increasing temperature.

Time for the drum to become empty = ____________ minutes

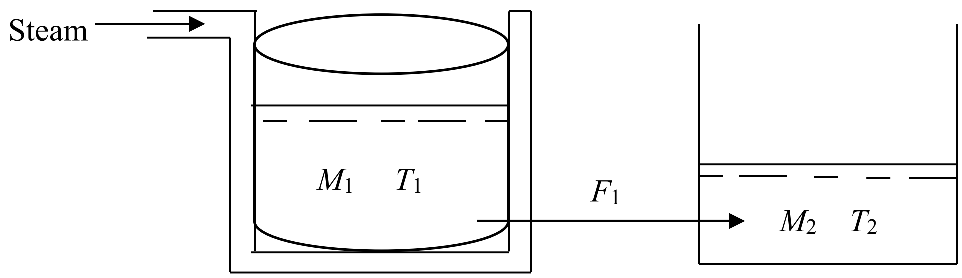
Does the pressure go through a maximum? Check one: Yes _____ No _____

If yes, P_{max} = ____________ atm at t = ____________ minutes

Time (min)	Pressure (atm)	Temp (Kelvin)
5		
10		
15		
20		

19. *Mass and Energy Balance in a Two-Tank System*

Consider the following two-tank system in which water in the first vessel (vertical cylindrical tank with a radius of R flows into the second one. Water in the first vessel is being heated with steam which enters a jacket surrounding the side walls and bottom of the cylindrical vessel. There is no heating in the second vessel. We define M_1, T_1, M_2, and T_2 as the mass and the temperature of water in the first vessel and in the second vessel, respectively.



(a) Using the following data about the system, derive an analytical expression for T_1 as a function of time. The heat transfer area in the first vessel cannot be assumed constant.

F_1 = 1.0 kg/min	U = 10.0 kcal/m^2-min-°C	T_{Steam} = 100 °C
C_P (water) = 1.0 kcal/kg-°C	ϕ (water) = 1000 kg/m^3	R = 0.2 m

Initially at $t = 0$: M_1 = 20 kg of water, T_1 = 20 °C, M_2 = 10 kg of water, T_2 = 80°C

Answer the following questions:

$T_1(t = 5 \text{ minutes})$ = ____________ °C $T_1(t = 10 \text{ minutes})$ = ____________ °C

(b) Notice that the water in the second vessel is very hot and its temperature will go down as soon as the cooler water from the first vessel enters the second vessel. However, this temperature decrease is only temporary and T_2 will eventually climb up again. Use MATLAB (*ode45*) to determine the following:

(i) The time (accurate to 2 decimal places) at which $T_1 = T_2$.
(ii) The time (accurate to 2 decimal places) at which T_2 goes through a minimum.

Answer the following questions (all answers in 2 decimal places):

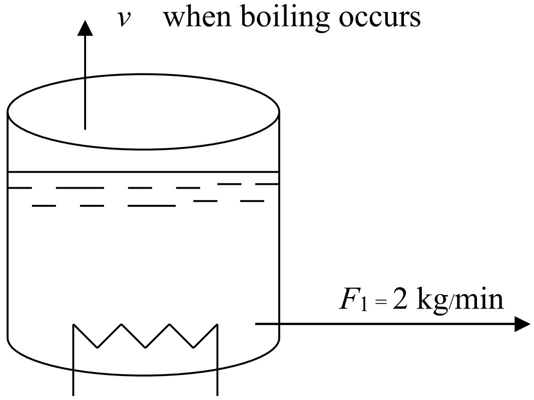
Time when $T_1 = T_2$ = ____________ minutes $T_1 = T_2$ = ____________ °C

Time when T_2 is minimum = ____________ minutes

T_2 (minimum) = ____________ °C

20. ***Boiling of Ethanol in a Cylindrical Vessel***

Consider the following cylindrical vessel which is initially filled with 100 kg of ethanol at the temperature of 20 °C. The vessel is constantly being heated with a heating coil (assume constant heat transfer area) at T = 100 °C and UA = 10 kJ/min-°C. The heating goes on until ethanol starts to boil at its normal boiling point of 78.55 °C. While the heating is taking place and then later when boiling occurs as well, ethanol also flows out of the vessel at the rate of F_1. Use the following additional data about ethanol to calculate the time it takes for ethanol to completely disappear from the cylindrical vessel. Also, how much faster does the boiling help in emptying the vessel?



MW (molecular weight) = 46.07 λ (heat of vaporization) = 38600 kJ/kmol

C_P (heat capacity) = 78.28 kJ/kmol-°C

Hint: Watch the units and their conversions carefully in your calculations

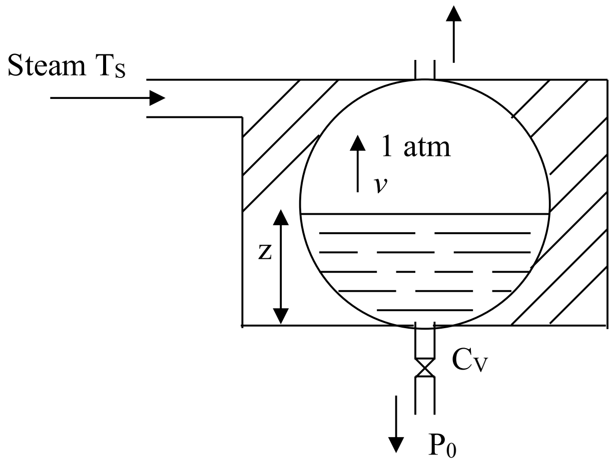
Answer the following questions:

Time when ethanol disappears completely from the vessel = ____________ minutes

Time by which the boiling helps the vessel to empty faster = ____________ minute

21. ***Boiling and Draining of Ethanol in a Spherical Vessel***

Consider the boiling of pure ethanol in a jacketed spherical vessel with a radius R = 1 meter, as shown in the diagram. The liquid is drained at the bottom of the vessel while some of it is boiled and escapes as vapor through the top of the vessel.



The following data are available:

$MW = 46.07$ $\lambda(T_B) = 3.858\text{x}10^7$ J/kmol
$\rho = 16.575$ kmol/m^3 $C_P = 1.4682\text{x}10^5$ J/kmol-°C

$$\log_{10} P^{vap} = 8.04494 - \frac{1554.30}{T + 222.65}$$ T in °C and P^{vap} in mmHg

$C_V = 4.5$ m^3-atm$^{-1/2}$/hr = valve constant
$T_S = 100$ °C $U = 2.0\times10^6$ J/hr-m^2-°C
$P_0 = 1$ atm $z_0 = z(t{=}0) = 1$ m

(a) Model this operation and use MATLAB to determine the time it takes for the vessel to completely empty, assuming that initially the liquid is at its boiling point. Note that the heat transfer area A_T is not constant.

(b) Repeat the calculations in Part (a), assuming that there is no draining of the liquid at the bottom (i.e. the liquid leaves the vessel only through boiling). Determine the solution analytically (an exact solution is possible in this case).

Useful conversion factors and formulae:

g = gravitational constant = 9.807 m/s^2

1 atm = $1.01325\text{x}10^5$ N/m^2 1 N = 1 kg-m/s^2

The volume of liquid V(z) in a spherical vessel as a function of its height z is given by

$$V(z) = \pi R z^2 - \frac{\pi z^3}{3}$$

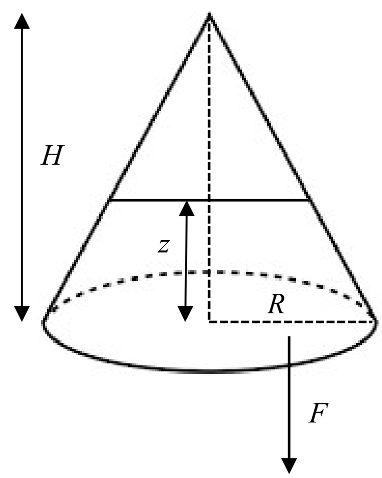
while the surface area S(z) is given by

$$S(z) = 2\pi R z$$

$$\int \frac{x}{a+bx}dx = \frac{[a+bx-a\ln(a+bx)]}{b^2}$$

22. *Boiling and Heat/Mass Balance in an Open Inverted Cone-Shaped Vessel*

An inverted cone-shaped vessel (R = 0.5 m, H = 2.0 m) is initially filled with pure water at the height of 1.0 m as measured from the bottom of the vessel (see the figure). The liquid in the vessel flows out from bottom with a rate F = 0.01 m^3/min.



The following symbols are defined for all parameters and variables in this system. Data are also given for all symbols considered as parameters.

V	=	Water volume inside the vessel (m^3)
ϕ	=	Mass density of water = 1000 kg/m^3
z	=	Liquid height inside the vessel as measured from the bottom (in m)
F	=	Outflow rate from the vessel = 0.01 m^3/min
C_P	=	Heat capacity of water = 1.0 kcal/kg-°C

The volume of a cone = $\frac{1}{3}\pi R^2 H$

Note that you can use this formula to derive the volume of the inverted cone at any given liquid height z. Be very careful with this derivation because the cone is inverted; otherwise all your subsequent answers will be wrong.

(a) Derive an analytical expression for z as a function of time, and answer the following question:

t when the vessel becomes empty = _______________ minutes

(b) Now, suppose the cone-shaped vessel is being heated via steam through the circular area at the bottom of the vessel (hence, the overall heat transfer area can be assumed to be constant). With the additional data given below, derive an analytical expression for T as a function of time (valid up till the boiling-point temperature of water at 100 °C). Then taking into account the vaporization of water, calculate the total time it takes for the vessel to become empty without using MATLAB. Hint: note that this total time must be less than the time calculated in Part (a).

T	=	Temperature of water inside the vessel (in °C) = 20 °C at $t = 0$.
λ	=	Heat of vaporization of water = 539.4 kcal/kg at 100 °C
U	=	Overall heat transfer coefficient = 5.0 kcal/m^2-min-°C
T_S	=	Steam temperature = 150 °C
v	=	Vaporization or boiling rate of water from the vessel (in kg/min)

t when the temperature in the vessel reaches 100 °C = _______________ minutes

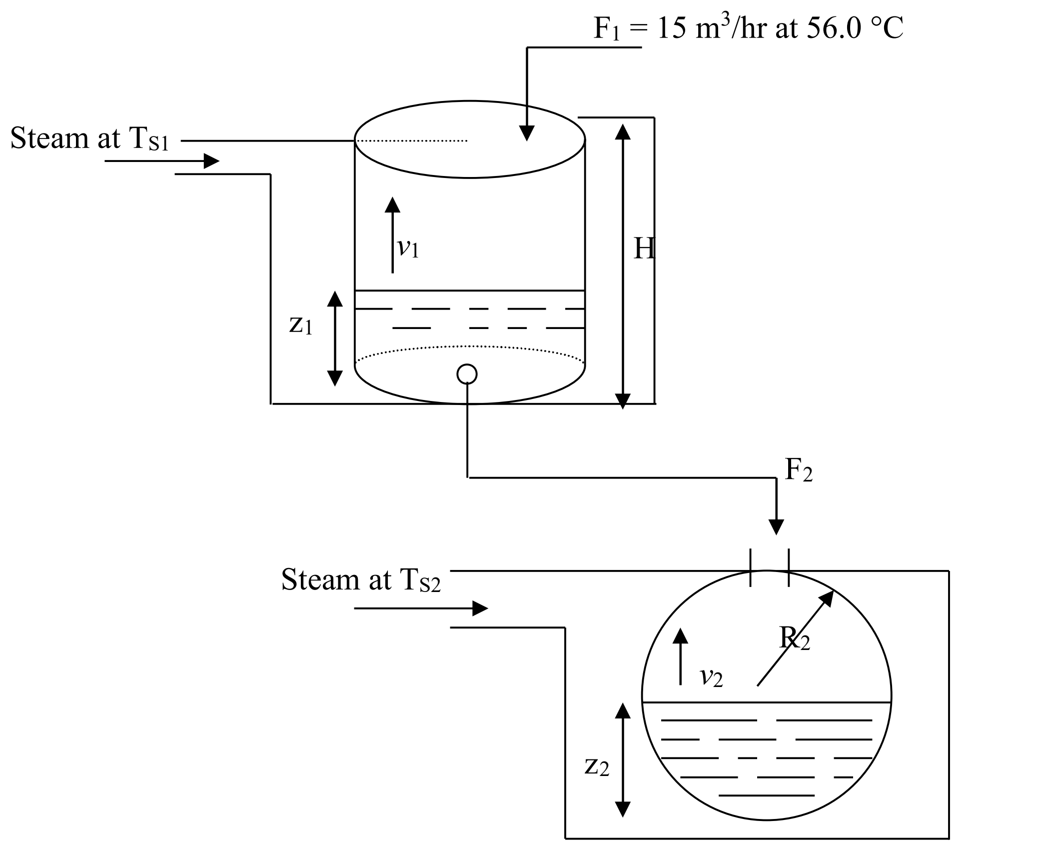
Total t when the vessel becomes empty = _______________ minutes

23. *Boiling of Acetone in a Cylindrical and a Spherical Vessels in Series*

Consider the boiling of pure acetone in a jacketed cylindrical vessel and a jacketed spherical vessel in series, as shown in the diagram. The cylinder has a radius of $R_1 = 1.0$ m and a height of H = 1.5 m, while the sphere has a radius of $R_2 = 1.0$ m. In addition to the liquid already in the vessel, pure acetone at its boiling point is added continuously at a rate $F_1 = 15$ m^3/hr. The cylinder also has a hole at the bottom, which allows acetone to flow out and into the sphere according to the relation

$$F_2 = C_d A (2 g z_1)^{1/2} \qquad \text{(in m}^3\text{/hr)}$$

where A is the area of the hole, g is the acceleration due to gravity, and C_d is an experimentally determined value.



The following data are available:

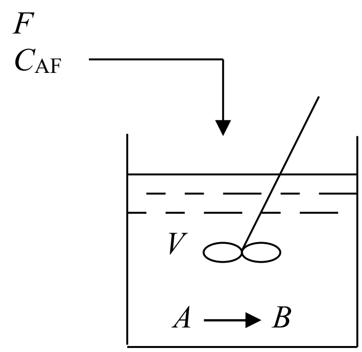
MW = 58.08	ρ = 13.62 kmol/m^3
$\lambda(T_B) = 3.02\text{x}10^7$ J/kmol	T_B = 56.0 °C
$C_P = 1.30735\text{x}10^5$ J/kmol-°C	U = 3.0x10^6 J/hr-m^2-°C (for both vessels)
T_{S1} = 150 °C	T_{S2} = 100 °C
$C_d = 0.5$	g = gravitational constant = 1.271x10^8 m/hr^2
r = hole radius = 0.01 m	
$z_1(t{=}0)$ = 0.5 m	$z_2(t{=}0)$ = 1.5 m

(a) Model this operation and determine the time in hours (correct to 2 decimal places) at which the cylindrical vessel will either empty completely or overflow. Assume that the liquid in both vessels is already at its boiling point at t = 0.

(b) Determine the time in hours (correct to 2 decimal places) at which the liquid height z_1 in the cylinder is equal to the liquid height z_2 in the sphere.

24. *Semi-Batch Reactor with a Single 1st-Order Reaction*

Consider the case where a batch reactor is being filled. Assume a single, first-order reaction ($A \rightarrow B$) and a sinusoidal volumetric flowrate into the reactor $F = 4.0 + 2\sin(10t)$ with no flow out of the reactor.



If C_{AF} (feed concentration) = 5 mol/m^3, k = 1 hr^{-1}, with V = 2.0 m^3 and $C_A = 0$ at $t = 0$, simulate the concentration of A as a function of time using MATLAB. Run the model for 10 hours and plot the concentration profile.

25. *Batch Reactor with a Series Reaction*

Consider the series reaction:

$$A \overset{k_1}{\rightarrow} B \overset{k_2}{\rightarrow} C \overset{k_3}{\rightarrow} D$$

(a) Assuming that each of the reactions is first-order and constant volume, write down the modeling equations for C_A, C_B, and C_C, where C_A, C_B, and C_C represent the concentrations (mol/volume) of components A, B, and C, respectively.

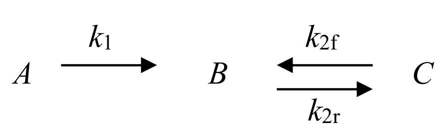
(b) Derive a third-order ODE for the concentration of C. That is, your ODE should look like

$$f\left(\frac{d^3C_C}{dt^3}, \frac{d^2C_C}{dt^2}, \frac{dC_C}{dt}, C_C, k_1, k_2, k_3\right) = 0$$

(c) If k_1 = 1 hr^{-1}, k_2 = 2 hr^{-1}, k_3 = 3 hr^{-1}, and $C_{A0} = C_A(t = 0)$ = 1 mol/liter, solve the ODE in Part (b) analytically for the concentration of C at any given time t.

26. *Batch Reactor with a Series Reaction*

Consider a batch reactor with a series reaction where component A reacts to form component B. Component B can also react reversibly to form component C. The reaction scheme can be characterized by:



Here k_{2f} and k_{2r} represent the kinetic rate constants for the forward and reverse reactions for the conversion of B to C, while k_1 represents the rate constant for the conversion of A to B.

(a) Assuming that each of the reactions is first-order and constant volume, write down the 3 modeling equations for C_A, C_B, and C_C, where C_A, C_B, and C_C represent the concentrations (mol/volume) of components A, B, and C, respectively.

(b) Using the following definitions:

Dimensionless time,	$\tau = k_1 t$
Conversion of A,	$x_1 = (C_{A0} - C_A) / C_{A0}$
Dimensionless concentration of B,	$x_2 = C_B / C_{A0}$
Ratio of rate constants,	$\alpha = k_{2f} / k_1$
Ratio of forward and reverse rate constants,	$\beta = k_{2r} / k_1$

Derive a second-order ODE for the dimensionless concentration of B. Your ODE must contain only dimensionless quantities (x_2, τ, α, and β).

(c) Solve the ODE in Part (b) analytically to find x_2 as a function of τ, α, and β.

(d) Using the following data:

$k_1 = 1.0 \ \text{min}^{-1}$ $\quad k_{2f} = 1.5 \ \text{min}^{-1}$ $\quad k_{2r} = 2.0 \ \text{min}^{-1}$
$C_{A0} = C_A(t = 0) = 3.0$ mol/liter
$C_{B0} = C_B(t=0) = 0$ mol/liter
$C_{C0} = C_C(t=0) = 0$ mol/liter

Solve for C_B analytically as a function of time.

(e) Given the data in Part (d), find the maximum concentration of B and the reaction time at this concentration. If no such maximum exists, prove it mathematically. Repeat the above calculations for the case of $k_1 = 3 \ \text{min}^{-1}$, $k_{2f} = 1.5 \ \text{min}^{-1}$, $k_{2r} = 1 \ \text{min}^{-1}$ while the initial conditions remain the same.

(f) Validate your analytical solutions by solving the differential equations in Part (a) with MATLAB and plot the time profiles of components A, B, and C. Use both sets of rate constants (i.e. $k_1 = 1.0 \ \text{min}^{-1}$, $k_{2f} = 1.5 \ \text{min}^{-1}$, $k_{2r} = 2.0 \ \text{min}^{-1}$ and $k_1 = 3.0 \ \text{min}^{-1}$, $k_{2f} = 1.5 \ \text{min}^{-1}$, $k_{2r} = 1.0 \ \text{min}^{-1}$).

27. *Modeling Two Reactions in Series in a Batch Reactor*

Consider the following two reactions in series:

$$A \overset{k_1}{\rightarrow} B \overset{k_2}{\rightarrow} C$$

The orders of the two reactions do not follow the stoichiometry but instead can be described by the following modeling equations. i.e. the first reaction is 3rd-order while the second reaction is 0th-order:

$$\frac{dC_A}{dt} = -k_1 C_A^3 \qquad \frac{dC_B}{dt} = k_1 C_A^3 - k_2 \qquad \frac{dC_C}{dt} = k_2$$

(a) Derive an exact (analytical) expression of C_A as a function of time in terms of k_1, given that at $t = 0$, $C_A = C_{A0}$, and $C_B = 0$. Then use the analytical solution to determine C_A at $t =$ 1.0 hour if $k_1 = 2.0$ (gmol/liter)$^{-2}$-hour^{-1} and $C_{A0} = 2.0$ gmol/liter and $C_{B0} = 0$.

C_A (t = 1.0 hour) = _______________ gmol/liter

(b) Derive an exact (analytical) expression of C_B as a function of time in terms of k_1 and k_2, given that at $t = 0$, $C_A = C_{A0}$, and $C_B = 0$. Simplify your final expression as much as possible. Then use the analytical solution to determine C_B at $t = 1$ hour, given that $k_1 = 2.0$ (gmol/liter)$^{-2}$-hour^{-1}, $k_2 = 0.5$ gmol/liter-hour, $C_{A0} = 2.0$ gmol/liter, and $C_{B0} = 0$.

C_B (t = 1.0 hour) = _______________ gmol/liter

(c) Calculate $C_{B,max}$ and the time at which C_B is at maximum based on the exact solution in Part (b).

$C_{B,max}$ (t = __________ hour) = _______________ gmol/liter

(d) Suppose the two reactions in series now follow the following modeling equations. Repeat Part (a).

$$\frac{dC_A}{dt} = -k_1\sqrt{C_A} \qquad \frac{dC_B}{dt} = k_1\sqrt{C_A} - k_2 C_B \qquad \frac{dC_C}{dt} = k_2 C_B$$

C_A (t = 1.0 hour) = _______________ gmol/liter

(e) Repeat Part (b) with the new modeling equations.

C_B (t = 1.0 hour) = _______________ gmol/liter

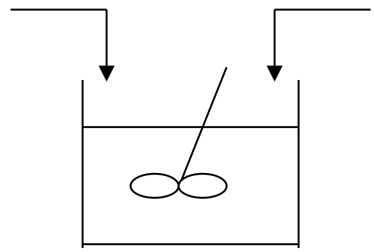
(f) Repeat Part (c) with the new modeling equations.

$C_{B,max}$ (t = __________ hour) = _______________ gmol/liter

28. *Isothermal Semi-Batch Reactor*

(a) Consider an isothermal semi-batch reactor where a single reaction takes place in a solvent *S*, which is inert. In this reaction, 2 moles of component *A* react with one mole of component *B* to form one mole of component *C*: $2A + B \xrightarrow{k_1} C$. The reaction rate does not conform to the stoichiometry but is 1st-order with respect to each reactant as follows: $r_A = -k_1 C_A C_B$

F_A = 15 liter/min $\qquad$ F_B = 10 liter/min



Initially (t = 0 min), the reactor contains 100 liters of solution and 300 moles of *A*. Assuming that all components have the same density of 60 mol/liter, derive 3 ODE equations needed to compute C_A, C_B, and C_C, the concentrations (moles/liter) of *A*, *B*, and *C*, respectively. Use *ode45* in MATLAB to solve for and plot (in a single graph) the concentrations of the 3 components. Run the model for 20 minutes with an increment of 0.5 minute.

The following experimental data have been obtained for this reaction when carried out in a batch reactor:

Time(minute)	0	5.0	7.5	12.0	15.5	25.0	32.0	40.0
C_A(mol/liter)	2.0	1.65	1.52	1.40	1.28	1.17	1.10	1.06
C_B(mol/liter)	0.5							

(b) At the end of 20 minutes, the 2 feeds to the reactor are suddenly shut off, and 4,500 moles of a new component called *D* (same density as components *A*, *B*, and *C*) are charged to the reactor. That is, the reactor now operates in a batch mode. Component *D* reacts with component *C* to form *A* and *B*, and the reaction now looks as follows:

$$2A + B \underset{k_2}{\overset{k_1}{\rightleftharpoons}} C + D$$

with a reaction rate of $r_A = -k_1 C_A C_B + k_2 C_D$ (2nd -order forward and 1st-order reverse). The value of k_2 has been measured to be 1.0 min^{-1}. Derive analytically the concentration of *A* as a function of time, and compute C_A at steady state based on your derived equation. Your final expression should be simplified as much as possible and should not contain any parameters except t (time) and C_A.

<u>Useful Integrals</u>:

$$\int \frac{dx}{x(a+bx)} = -\left(\frac{1}{a}\right)\ln\left(\frac{a+bx}{x}\right)$$

$$\int \frac{dx}{B} = \begin{cases} +\dfrac{2}{\sqrt{\gamma}}\tan^{-1}\left(\dfrac{\omega}{\sqrt{\gamma}}\right) & if \quad \gamma > 0 \\ -\dfrac{2}{\omega} & if \quad \gamma = 0 \\ -\dfrac{2}{\sqrt{-\gamma}}\tanh^{-1}\left(\dfrac{\omega}{\sqrt{-\gamma}}\right) & if \quad \gamma < 0 \end{cases}$$

where $B = a + bx + cx^2$
$\gamma = 4ac - b^2$
$\omega = b + 2cx$

29. *Determination of Reaction Kinetics*

(a) The following laboratory data were obtained for the irreversible reaction under isothermal constant-volume conditions:

$$2A \rightarrow B + C$$

Time (min)	0	3.2	5.0	9.6	12.5	18.4	25.0
[A] (mol/liter)	0.1345	0.0772	0.0602	0.0352	0.0261	0.0168	0.0110

Determine the kinetics to explain these data (i.e. find the order of the reaction and its rate constant).

(b) The following laboratory data were obtained for the irreversible reaction under isothermal constant-volume conditions:

$$A + B \rightarrow C + D$$

Time (sec)	0	2790	7690	9690	14000	19100
[*A*] (mol/liter)	0.1908	0.1833	0.1745	0.1719	0.1682	0.1650
[*B*] (mol/liter)	0.0313	0.0238	0.0150	0.0123	0.0086	0.0055

Determine the kinetics to explain these data.

30. *Isothermal Batch Reactor with a Series Reaction*

Consider an isothermal batch reactor with a series reaction where 2 moles of component *A* react to form one mole of component *B*. Component *B* also reacts to form component *C*. The reaction scheme can be characterized as follows:

$$2A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

Initially (t = 0 hr), $C_A = C_{A0}$, $C_B = 0$, and $C_C = 0$, where C_A, C_B, and C_C represent the concentrations (mol/liter) of components *A*, *B*, and *C*, respectively.

(a) Assume constant volume and that the first reaction $2A \rightarrow B$ is one-half order and the second reaction $B \rightarrow C$ is first-order, i.e.

$$\frac{dC_A}{dt} = -k_1\sqrt{C_A}$$

$$\frac{dC_B}{dt} = \frac{k_1}{2}\sqrt{C_A} - k_2 C_B$$

Derive an analytical expression for C_B as a function of time.

(b) The following experimental data were obtained for component C_A:

Time(hr)	0	0.05	0.15	0.35	0.60	0.85	1.0
C_A(mol/liter)	1.0	0.93	0.79	0.54	0.30	0.13	0.06

C_B was also measured to be 0.11 mol/liter at t = 1 hr. Determine the values of k_1 and k_2 and the time t_{max} at which C_B is at its maximum. **Hint**: Use *Polyfit* function in MATLAB to help determine k_1 and k_2.

(c) Now, suppose the order of the above series reaction conforms to the stoichiometry, derive analytically a 1st-order ODE for C_B, i.e.

$$\frac{dC_B}{dt} + p(t)C_B = q(t)$$

but do not solve this ODE.

(d) When the reaction order conforms to the stoichiometry, the following experimental data were obtained for C_A:

Time(hr)	0	0.03	0.06	0.10	0.15	0.20	0.30
C_A(mol/liter)	1.0	0.76	0.63	0.51	0.39	0.33	0.25

It was also observed that C_B reached a maximum of 0.14 mol/liter at $t = 0.1$ hr. Determine the values of k_1 and k_2.

31. *Modeling an Isothermal Reversible Reaction and Determining Rate Constants*

Consider an isothermal batch reactor with the following reversible reaction:

$$A \underset{k_2}{\overset{k_1}{\rightleftarrows}} B$$

The reversible reaction does not follow the stoichiometry. Instead, the forward reaction is zeroth-order while the backward reaction is second-order with respect to the reactant, i.e.

$$\frac{dC_A}{dt} = -k_1 + k_2 C_B^2$$

$$\frac{dC_B}{dt} = k_1 - k_2 C_B^2$$

(a) Given that at $t = 0$, $C_A = C_{A0}$ and $C_B = C_{B0}$, derive analytically an expression of C_B as a function of time. Note that your final equation for C_B must be expressed in terms of C_{B0}, k_1, and k_2.

(b) Based on the result in Part (a), derive now an analytical expression of C_A as a function of time. Again, your final equation for C_A must be expressed in terms of C_{A0}, C_{B0}, k_1, and k_2.

(c) The following data are available from experiments:

t (hour)	0	0.05	0.15	0.30	0.40	0.55	0.70
C_B (mol/liter)	6.0	4.40	3.10	2.36	2.20	2.08	2.03
C_A (mol/liter)	6.0						

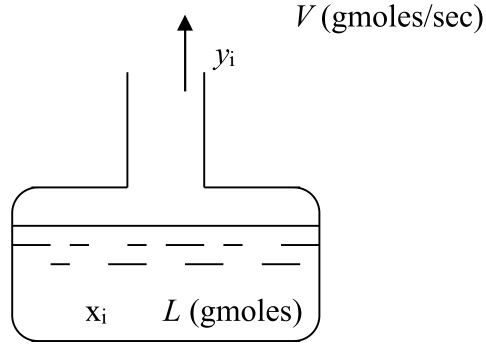
Also, it was observed that at steady-state, C_B = 2.0 mol/liter. Determine the values of the two rate constants k_1 and k_2.

Answer the following questions:

k_1 = ______________ mol/hr $\qquad$ k_2 = _____________ liter^2/mol-hr

32. *Batch Distillation, I*

A liquid mixture containing 70.0 mole% *n*-pentane and 30.0 mole% *n*-hexane is distilled in a batch still and is initially charged with 100 gmoles of the mixture.



(a) The equilibrium relationship between the mole fraction x of n-pentane in the liquid and that in the vapor y is of the form

$$y_{C5} = x_{C5}\left(a + \frac{b}{1 + x_{C5}^2}\right)$$

Note that this equation is only valid for *n*-pentane, and not necessarily for *n*-hexane. Assuming ideal gas and ideal liquid and given

For *n*-pentane, $\log_{10} P^{VAP} = 6.85221 - \dfrac{1064.630}{T + 232.00}$

For *n*-hexane, $\log_{10} P^{VAP} = 6.87776 - \dfrac{1171.53}{T + 224.366}$

P^{VAP} in mmHG
T in °C

Calculate the mole fraction of pentane in the vapor phase in equilibrium with the 70 mole% pentane-30 mole% hexane mixture at the initial system temperature of 46 °C. Also, calculate the coefficients *a* and *b* in the *n*-pentane *x*-*y* relationship.

(b) At any given instant, the vapor leaving the still may be considered to be in equilibrium with the remaining liquid. Assuming that the values of *a* and *b* do not change with time and that vapor and liquid phases are constantly in equilibrium with each other, derive an analytical equation relating *L*, the amount of liquid left in the still, to *x*, the mole fraction of *n*-pentane in this liquid, i.e.

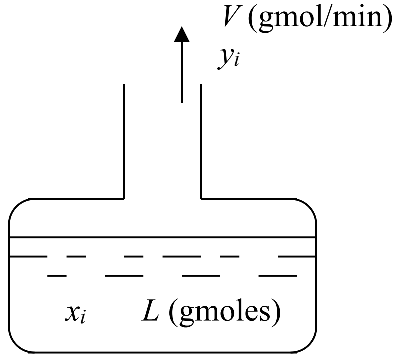
$$L = \text{function}(x_{C5})$$

33. *Batch Distillation, II*

A liquid mixture containing 60.0 mole% *n*-pentane and 40.0 mole% *n*-hexane is to be distilled in a batch still and is initially charged with 100 gmoles of the mixture. The equilibrium relationship between the mole fraction *x* of *n*-pentane in the liquid and that in the vapor *y* has been correlated to the following equation:

$$y = 1.8804\,x - 0.8804\,x^2$$

Note that this equation is only valid for *n*-pentane, and not necessarily for *n*-hexane.



(a) Calculate the system temperature at which the above equilibrium relationship was established for the given liquid mixture (i.e. determine *T* in °C at which the above equation is valid). Also, compute the total system pressure *P* at the system temperature.

Assume ideal gas and ideal liquid, and the vapor pressures of the two components are:

For *n*-pentane, $$\log_{10} P^{VAP} = 6.85221 - \frac{1064.630}{T + 232.00}$$

For *n*-hexane, $$\log_{10} P^{VAP} = 6.87776 - \frac{1171.53}{T + 224.366}$$

P^{VAP} in mmHG
T in °C

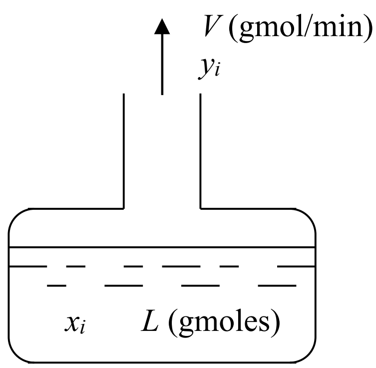
(b) Derive an analytical expression relating the amount of liquid left in the batch still L as a function of x (mole fraction of n-pentane in the still) and compute the value of x after 90% of liquid has been vaporized.

34. *Batch Distillation, III*

An equimolar liquid mixture of ethanol and water is to be distilled in a batch still at 1 atm and is initially charged with 100 gmoles of the mixture. Because water and ethanol forms a rather nonideal solution, Raoult's Law which fails to predict the azeotrope of the mixture cannot be used for vapor-liquid equilibrium calculations. Instead, the equilibrium relationship between the mole fraction x of ethanol in the liquid and that in the vapor y is correlated to the following empirical equation:

$$y = ax^3 + bx^2 + cx$$

where a, b, and c are constants, which can be determined from experimental data. Note that the empirical equilibrium equation must and does satisfy the two end-points of the x-y curve at $x = 0$ and $x = 1$. The following two pairs of data are known about the mixture: the azeotrope occurs at 89.43 mol% ethanol and when $x = 0.1661$, $y = 0.5089$.



Derive an analytical expression relating the amount of liquid left in the batch still L as a function of x (mole fraction of ethanol in the still).

<u>Answer the following questions</u>:

(i) Compute the value of x after 75% of liquid has been vaporized: x = ____________

(ii) How much liquid is left when $y = 0.70$? L = ____________ gmol

Carry four decimal places in your hand calculations. You may use MATLAB to solve any linear or nonlinear equations you encounter during your derivation of the analytical solution.

35. *Batch Distillation of a Water/Acetic-Acid System*

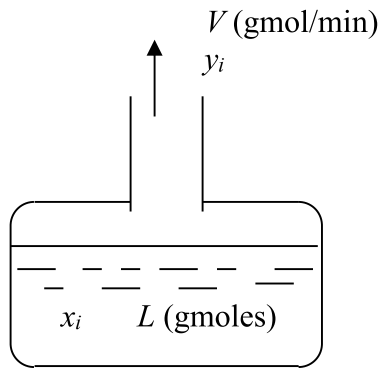
The following vapor-liquid equilibrium data (mole fractions) are available for the water/acetic-acid binary system at 25 °C.

x_{water}	y_{water}	x_{water}	y_{water}
0	0	0.6	0.6993
0.1	0.2329	0.7	0.7770
0.2	0.3558	0.8	0.8530
0.3	0.4517	0.9	0.9274
0.4	0.5379	1.0	1.0
0.5	0.6198		

First, we will use the *Polyfit* function in MATLAB to fit the above data to a quadratic polynomial in the form of

$$y = ax^2 + bx + c$$

The water-acid system is distilled in a batch still as shown, which is initially charged with 100 gmoles of the mixture. The heating of the batch still is controlled such that it vaporizes 5 gmoles of the liquid every minute.



It has also been observed that it takes exactly 10 minutes to distill the mixture to reach 50 mole% water in the liquid. Without using the *ode* solver in MATLAB (although you may use it to verify your answer), compute analytically the initial composition of the liquid mixture.

Answer the following question:

Initial mole fraction of water in the liquid mixture = _______________

Useful Integral:

$$\int \frac{dx}{B} = \begin{cases} +\dfrac{2}{\sqrt{\gamma}} \tan^{-1}\left(\dfrac{\omega}{\sqrt{\gamma}}\right) & if \quad \gamma > 0 \\ -\dfrac{2}{\omega} & if \quad \gamma = 0 \\ -\dfrac{2}{\sqrt{-\gamma}} \tanh^{-1}\left(\dfrac{\omega}{\sqrt{-\gamma}}\right) & if \quad \gamma < 0 \end{cases}$$

where $B = a + bx + cx^2$
$\gamma = 4ac - b^2$
$\omega = b + 2cx$

Part II

Basic Principles of Process Optimization

1. ***Solving Constrained Optimization Problems by Graphical Method, I***

Solve the following nonlinear constrained optimization problems by graphical method by finding their maxima and minima. After drawing your diagrams, you may need to use sophisticated algebra and calculus to determine the exact optima in each problem.

(a)
$$f(x,y) = x^2 + 4y^2$$
$$s.t.$$
$$y - x \geq 2$$
$$0 \leq x \leq 5$$
$$0 \leq y \leq 4$$

(b)
$$f(x,y) = (x-10)^2 + (y-6)^2$$
$$s.t.$$
$$y - 0.2x \leq 10$$
$$y - 0.1x^2 + 2x \geq 12$$

(c)
$$f(x,y) = x + y$$
$$s.t.$$
$$y \geq x^2$$
$$\frac{(y-2)^2}{36} + \frac{(x-1)^2}{9} \leq 1$$

2. ***Solving Constrained Optimization Problems by Graphical Method, II***

Solve the following nonlinear constrained optimization problems by graphical method by finding their maxima and minima. After drawing your diagrams, you may need to use sophisticated algebra and calculus to determine the exact optima in each problem.

(a)
$$f(x,y) = -x + 2y$$
$$s.t.$$
$$x^2 + y^2 \leq 5$$
$$x - y \geq -1$$

(b)
$$f(x_1,x_2) = x_1$$
$$s.t.\quad (x_1-2)^2 + (x_2-2)^2 \leq 9$$
$$x_1 - x_2 \leq 2$$

(c)
$$f(x_1,x_2) = x_1^2 + x_2^2$$
$$s.t.$$
$$x_1 - (x_2-2)^2 \geq -1$$
$$2x_1 - x_2 \leq 2$$

3. *Solving Constrained Optimization Problems by Graphical Method, III*

Solve the following two optimization problems by graphical method. After drawing your diagrams, use sophisticated algebra and calculus to determine the exact optima in each problem.

(a) Find both the minimum and maximum of:

$$f(x, y) = y - x$$

$$\begin{aligned} s.t. \quad y - 3x &\leq 2 \\ y + x &\leq 4 \\ -y + 0.5x &\leq 4 \\ x &\geq 0 \end{aligned}$$

(b) Find both the minimum and maximum of:

$$f(x, y) = y - x^2 + 2x$$

$$\begin{aligned} s.t. \quad y - x &\leq 2 \\ y + x &\leq 8 \\ y - 0.2x &\geq -2 \end{aligned}$$

Calculate the optimal objective values in both problems as well.

Instructions: For Problems 4 through 21, classify each problem as either linear (LP), quadratic (QP), nonlinear (NLP), integer linear (ILP), or mixed integer linear (MILP) programming problem. If the problem is quadratic or nonlinear, state whether it is constrained. Some problems can be solved by calculus but some cannot. Do not solve each problem unless you are explicitly told to do so.

4. *Optimal Height and Diameter of an Absorption Tower*

An absorption tower containing wooden grids is to be used for absorbing SO_2 in a sodium sulfite solution. A mixture of air and SO_2 will enter the tower at a rate of 70,000 ft^3/min, temperature of 250°F, and pressure of 1.1 atm. The concentration of SO_2 in the entering gas is specified, and a given fraction of the entering SO_2 must be removed in the absorption tower. The molecular weight of the entering gas mixture may be assumed to be 29.1. Under the specified design conditions, the number of transfer units necessary varies with the superficial gas velocity as follows:

$$\text{Number of transfer units} = 0.32G_s^{0.18}$$

where G_s is the entering gas velocity as lbm/(hr-ft^2) based on the cross-sectional area of the empty tower. The height of a transfer unit is constant at 15 ft. The cost for the installed tower is $1 per cubic foot of inside volume, and annual fixed charges amount to 20 percent of the initial cost. Variable operating charges for the absorbent, blower power, and pumping power are represented by the following equation:

$$\text{Total variable operating cost as \$/hr} = 1.8 \times 10^{-8} G_s^2 + \frac{81}{G_s} + \frac{4.8}{G_s^{0.8}}$$

The unit is to operate 8,000 hr/year. Formulate this problem as an optimization problem to minimize the annual cost and classify it.

You must simplify all your equations so that in the end the only variable(s) left in the formulation are the unknowns (decision variables).

5. *Formulation of a Manufacturing Optimization Problem*

A chemical manufacturing firm has discontinued production of a certain unprofitable product line. This has created considerable excess production capacity on the three existing batch production facilities. Management is considering devoting this excess capacity to one or more of three new products: call them products 1, 2, and 3. The available capacity on the existing units which might limit output is summarized in the following table, and each of the three new products requires the following processing time for completion:

Unit	**Available Time,** hours/week	**Productivity,** hours/batch		
		Product 1	**Product 2**	**Product 3**
A	20	0.8	0.2	0.3
B	10	0.4	0.3	…..
C	5	0.2	…..	0.1

The sales department indicates that the sales potential for products 1 and 2 exceeds the maximum production rate and that the sales potential for product 3 is 20 batches per week. The profit per batch would be $20, $6, and $8, respectively, on products 1, 2, and 3.

Formulate this problem as an optimization problem and classify it. Assume that the number of batches needs not be integer. Propose three feasible solutions (for example, one solution may involve the production of just product 1 and no product 2 and 3). Compare the objective function of your three feasible solutions.

6. *Maximizing Total Profit in Making Two Display Boxes*

A cabinet maker is asked by a jeweler to build two display boxes with lids made from a special shatter and break-proof transparent material. Only 8 square feet of the material are in stock at present and no more can be procured within the time available to build the boxes.

For Box 1, the length has to be 1 foot whereas the width and height are to be equal to an unspecified size x_1. For Box 2, the height has to be 1 foot whereas the width and length are to be equal to unspecified size x_2.

The cost of the material and labor is $ 0.5 per square foot of the outside of each box (Hint: There are 6 sides to each box!). The cabinet maker charged the jeweler proportional to the sum of the three dimensions of the boxes, i.e., width + length + height at the rate of $ 2.00 for box 1 and $ 2.50 per foot for Box 2. The cabinet maker's problem is to determine nonnegative values of x_1 and x_2 which will satisfy the material constraint on the lids of the two boxes and at the same time maximize his total profit.
Formulate the cabinet maker's problem as an optimization problem and classify it.

7. *Formulation of a Manufacturing Optimization Problem*

A certain plant can manufacture 5 different products in any combination. Each product requires time on each of three machines in the following manner (all numbers are in minutes/unit):

Product	Machine		
	1	2	3
A	12	8	5
B	7	9	10
C	8	4	7
D	10	0	3
E	7	11	2

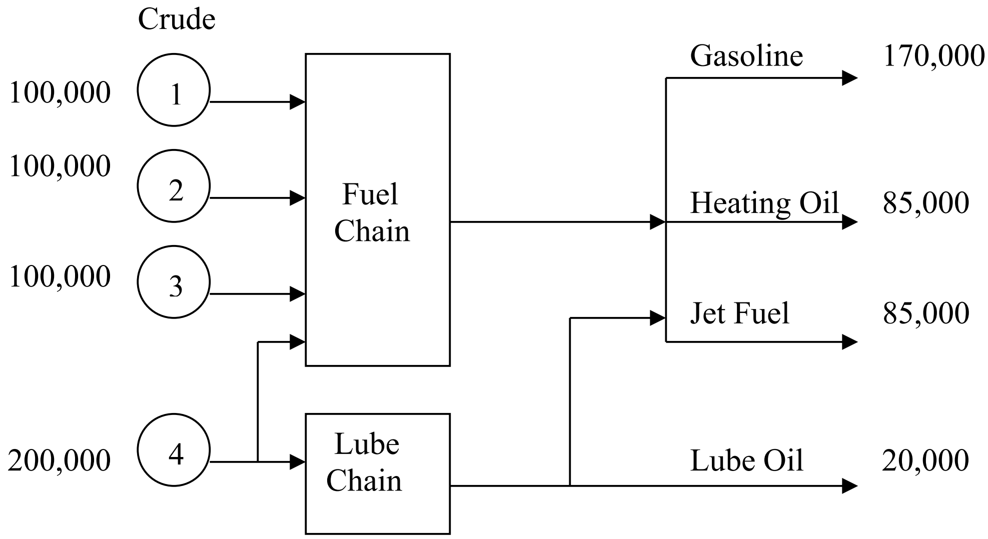
Each machine is available 128 hours per week. Products *A*, *B*, and *C* are purely competitive, and any amounts made may be sold at respective prices of $5, $4, and $5. The first 20 units of *D* and *E* produced per week can be sold at $4 each, but all made in excess of 20 can only be sold at $3 each. Variable labor costs are $4 per hour for machines 1 and 2 and $3 per hour for machine 3. Material costs are $2 for products *A* and *C*, and $1 for products *B*, *D*, and *E*. You wish to maximize profit to the firm. Formulate this problem as an optimization problem and classify it.

8. *Optimization of a Refinery Production*

A refinery has four different crudes which are to be processed to yield four products: gasoline, heating oil, jet fuel, and lube oil. There are maximum limits both on product demand (what can be sold) and crude availability. A schematic of the processing operation is as follows:

Crude Availability
bbl/wk = Barrels/week

Max. Product Demand
bbl/wk



(a) Given the tabulated profits, costs, and yields in Table 1, formulate the problem as an optimization problem that maximizes the profit and classify it.

(b) Propose 3 feasible solutions (which are not necessarily optimal) to this problem.

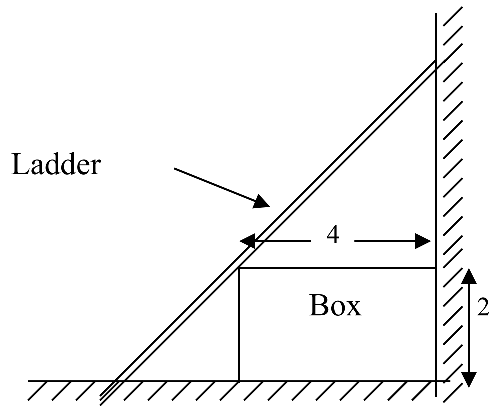
Table 1: Profits, Costs, and Yields Data

		Crude Type					
		1	**2**	**3**	**4** Fuel Process	**4** Lube Process	**Product Value** $/bbl
Yields	Gasoline	0.6	0.5	0.3	0.4	0.4	45.00
(bbl product	Heating Oil	0.2	0.2	0.3	0.3	0.1	30.00
per bbl crude)	Jet Fuel	0.1	0.2	0.3	0.2	0.2	15.00
	Lube Oil	0	0	0	0	0.2	60.00
	Other*	0.1	0.1	0.1	0.1	0.1	
Crude cost $/bbl		15.00	15.00	15.00	25.00	25.00	
Operating cost $/bbl		5.00	8.50	7.50	3.00	2.50	

* “Other” refers to losses in processing

9. ***Finding the Shortest Length of a Ladder***

A rectangular box of height 2 meters and width 4 meters is placed adjacent to a wall (see figure below). Formulate this problem as an optimization and classify it. Find the length of the shortest ladder that can be made to lean against the wall.



10. ***Formulation and Minimization of a Heat Transfer Problem***

It is desired to cool a gas [C_P = 0.3 Btu/lbm-°F] from 195 to 90°F, using cooling water [C_P = 1.0 Btu/lbm-°F, ρ = 62.4 lbm/ft^3] at 80°F in a counter-current heat exchanger. Water costs \$0.20/1000 ft^3, and the annual fixed charges for the exchanger are \$0.50/ft^2 of heat transfer area. The heat transfer coefficient is U = 8 Btu/hr-ft^2-°F for a gas rate of 3000 lbm/hr. The heat exchanger is operated 365 days/year (24 hours/day).

Formulate this heat exchange operation as an optimization problem to minimize the total annual cost and classify it. Also, use calculus to solve for the optimum by eliminating the constraints and transforming the problem into an unconstrained problem.

Hint: ΔT_{lm} = log-mean $\Delta T = (\Delta T_2 - \Delta T_1)/\ln(\Delta T_2/\Delta T_1)$
where ΔT_2 and ΔT_1 are the terminal temperature differences of heat exchanger.

11. ***Optimum Thickness of Insulation for a Furnace***

To reduce heat losses, the exterior flat wall of a furnace is to be insulated. To determine the optimum insulation thickness, it is necessary to consider and balance the costs of the insulation and the value of the energy saved by adding the insulation. The rate of heat transfer Q through the wall is:

$$Q = UA\,(T_{\text{furnace}} - T_{\text{wall}})$$

where T is in °F and Q is in Btu/hr. The overall heat transfer coefficient U is related to the outside convective heat transfer coefficient h and the thermal conductivity of insulation k by:

$$\frac{1}{U} = \frac{1}{h} + \frac{t}{12k}$$

where t is the thickness in inches of the insulation.

We wish to maximize the savings in total operating cost, savings expressed as the difference between the dollar value of the heat conserved minus the cost of the insulation over a span of 5 years (after that, the insulation will have to be replaced). Formulate this problem as an optimization problem, classify it, and obtain the optimum value of t (in inches) using the following data:

Temperature inside the furnace	500°F (constant)
Air temperature outside wall	Assume constant at 70°F
Heat transfer coefficients	
Outside air film h	4.0 Btu/(hr)(ft^2)(°F)
Conductivity of insulation k	0.03 Btu/(hr)(ft)(°F)
Total cost of insulation (per unit area per inch of thickness)	\$0.75/(ft^2)(per inch of thickness)
Values of energy saved	\$0.60/10^6 Btu
(i.e. the dollar difference between adding insulation and having no insulation for every 1 million Btu is \$0.60.)	
Hours of operation	8700 hours/year

Prove that your optimum t is a maximum. **Hint:** The heat transfer area A is constant.

12. *An Airline Assignment Problem*

Large airlines tend to base their route structure around the hub concept. An airline will try to have a large number of flights arrive at the hub airport during a certain short interval of time, e.g. 9 A.M. to 10 A.M. and then have a large number of flights depart the hub shortly thereafter, e.g. 10 A.M. to 11 A.M. This allows customers of that airline to travel between a large combination of origin/destination cities with one stop and at most one change of planes. For example, United Airlines uses Chicago as a hub, Delta Airlines uses Atlanta, TWA uses St. Louis, and American uses Dallas/Fort Worth. A desirable goal in using a hub structure is to minimize the amount of changing of planes (and the resulting moving of baggage) at the hub. The following problem illustrates this airline assignment problem.

A certain airline has 6 flights arriving at O'Hare airport between 9 and 9:30 A.M. The same 6 airplanes depart on different flights between 9:40 and 10:20 A.M. The average numbers of people transferring between incoming and leaving flights appear in the table below:

	L01	L02	L03	L04	L05	L06
I01	20	15	16	5	4	7
I02	17	15	33	12	8	6
I03	9	12	18	16	30	13
I04	12	8	11	27	19	14
I05	0	7	10	21	10	32
I06	0	0	0	6	11	13

Flights I05 arrives too late to connect with L01. Similarly, I06 is too late for flights L01, L02, and L03.

All the planes are identical. A decision problem is which incoming flight should be assigned to which outgoing flight. For example, if incoming flight 102 is assigned to leaving flight L03, 33 people (and their baggage) will be to remain on their plane at the stop at O'Hare. Formulate this assignment problem as an optimization and classify it.

13. *Toll-Way Staffing Problem*

The Northeast Tollway out of Chicago has a toll plaza with the following staffing demands during each 24-hour period:

Hours	Collectors Needed
12 a.m. to 6 a.m.	2
6 a.m. to 10 a.m.	8
10 a.m. to 12 a.m.	4
12 a.m. to 4 p.m.	3
4 p.m. to 6 p.m.	6
6 p.m. to 10 p.m.	5
10 p.m. to 12 Midnight	3

Each collector works 4 hours, is off one hour, and then works another 4-hours. A collector can be started at any hour. Assuming the objective is to minimize the number of collectors hired, formulate this problem as an optimization problem.

14. *Formulation of a Manufacturing Problem*

A chemical company manufactures four products (1, 2, 3, and 4) on two machines (*X* and *Y*). The time (in minutes) to process one unit of each product on each machine is shown below:

		Machine *X*	Machine *Y*
Product	1	10	27
	2	12	19
	3	13	33
	4	8	23

The profit per unit for each product (1, 2, 3, and 4) is \$10, \$12, \$17, and \$8 respectively. Each product can be produced on either machine.

The factory is very small and this means that floor space is very limited. Only one week's production is stored in 50 square meters of floor space where the floor space taken up by each product is 0.1, 0.15, 0.5, and 0.05 (square meters) for products 1, 2, 3, and 4 respectively.

Customer requirements mean that the amount of product 3 produced should be related to the amount of product 2 produced. Over a week, twice as many units of product 2 should be produced as product 3.

Machine *X* is out of action (for maintenance/because of breakdown) 5% of the time and machine *Y* 7% of the time.

Assuming a working week 35 hours long, formulate the problem of how to manufacture these products as a linear program so as to maximize the total weekly profit and classify it.

15. *Maximizing the Monetary Value of a Cargo Load*

A cargo load is to be prepared from five types of articles. The weight w_i, volume v_i, and monetary value c_i of different types of articles are given below.

Article Type	w_i	v_i	c_i
1	4	9	5
2	8	7	6
3	2	4	3
4	5	3	2
5	3	8	8

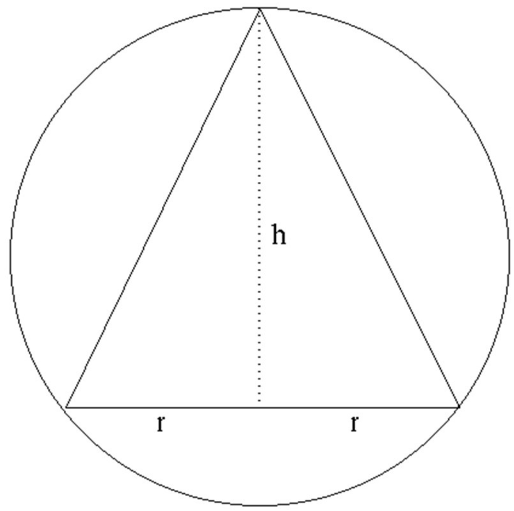
Formulate this optimization problem which will determine the number of articles x_i selected from the i^{th} type (i = 1, 2, 3, 4, 5), so that the total monetary value of the cargo load is maximized. The total weight and volume of the cargo cannot exceed the limits of 2000 and 2500 units, respectively. Classify the problem but do not solve for the optimum. Note that it is not possible to load a fraction of an article.

16. *Formulating and Solving a Vessel-Design Optimization Problem*

As a starting chemical engineer at Vessels Fabrication Co., Ltd., your first assignment is to design a cone-shaped vessel. Specifically, your job is to calculate the dimensions (radius *r* and height *h*) of the cone with a maximum volume that can be inscribed (fit into) in a sphere of radius 2 meters. Formulate this assignment as an optimization problem, classify it, and solve it. Note that the volume of a cone is given by

$$V = \frac{1}{3}\pi \ r^2 h$$

Also, show that the volume found is indeed a maximum. A two-dimensional side view of the geometry is shown below.



two–
dimensional
side view

17. ***Formulation of a Cargo Plane Optimization Problem***

A cargo plane has three compartments for storing cargo: front, center, and rear. These compartments have the following limits on both weight and space:

Compartment	Weight capacity (tons)	Space capacity (m^3)
Front	10	6800
Center	16	8700
Rear	8	5300

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the plane. The following four cargoes are available for shipment on the next flight:

Cargo	Weight (tons)	Volume (m^3/ton)	Profit ($/ton)
$C1$	18	480	310
$C2$	15	650	380
$C3$	23	580	350
$C4$	12	390	285

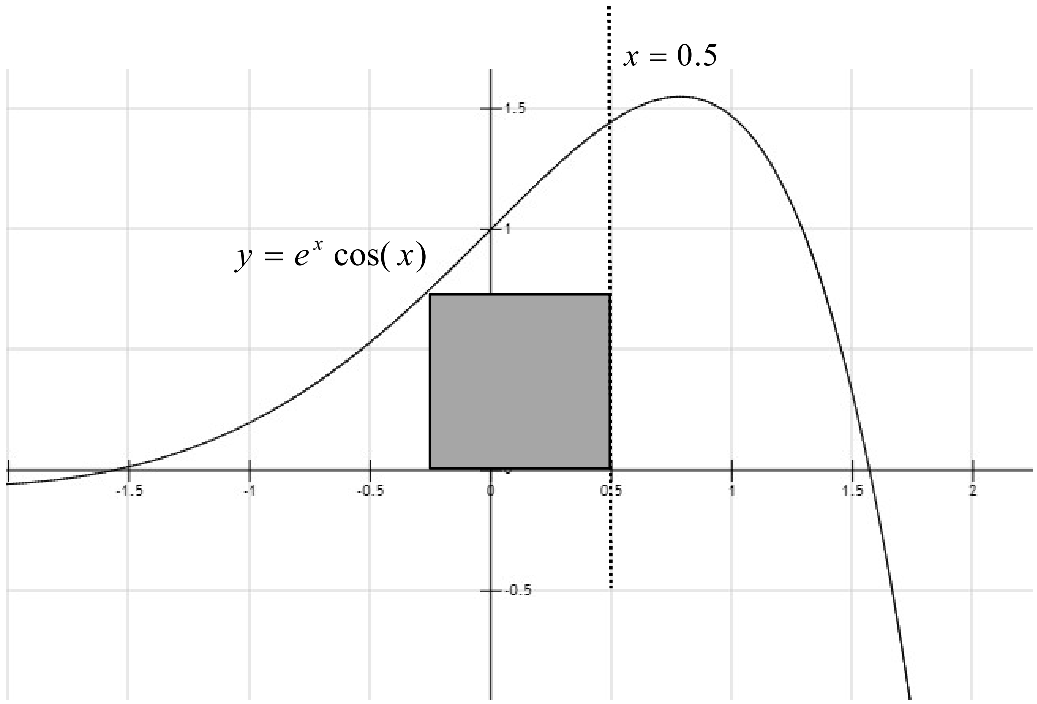
Any proportion of these cargoes can be accepted. The objective is to determine *how much* (if any) of each cargo $C1$, $C2$, $C3$ and $C4$ should be accepted and *how to distribute* each among the compartments so that the total profit for the flight is maximized. Formulate the above optimization problem and classify it, but do not solve it.

18. ***Minimum Area under a Curve and Shortest Distances to the Curve***

The graph below shows the function $y = \exp(x)\cos(x)$ over the domain $[-2, 2]$.

(a) A shaded rectangle is enclosed under the curve between the curve, the vertical line $x = 0.5$, and the x-axis as shown. Calculate the maximum area of the shaded rectangle that is possible under the curve. Also, calculate the total unshaded area under the curve when the shaded rectangle has the maximum area.

Recall the formula for integration by parts: $\int u\,dv = uv - \int v\,du,$



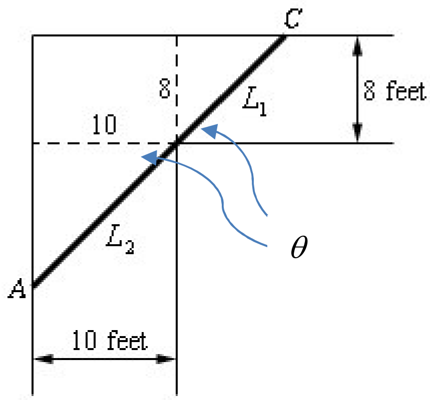
(b) Determine the point on the curve which has the shortest distance to the coordinate [0.5, 1.0].

19. ***Solving Optimization Problems by Calculus***

As a chemical engineer, you have been called upon to solve the following optimization problems:

(a) A long piece of pipe is being carried down a hallway that is 10 feet wide. At the end of the hallway there is a right-angled turn and the hallway narrows down to 8 feet wide. What is the longest pipe that can be carried (always keeping it horizontal) around the turn in the hallway (i.e. the minimum pipe length that will fit around the turn and anything larger will not fit)? Prove that calculated pipe length is the minimum. Hnt:

Rather than using Pythagorean Theorem which results in complicated math, use trigonometry involving the angle θ to solve this optimization problem.



Useful Formula:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$$

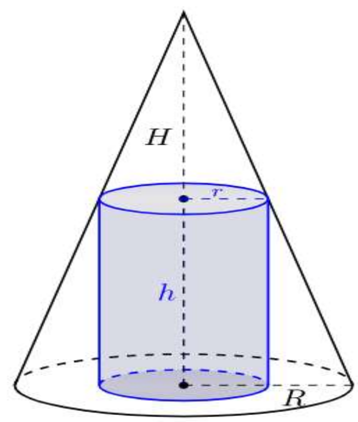
Answers: L_1 = _______________ ft; L_2 = _______________ ft

(b) You are asked to design a special piece of equipment which is a cone in which a cylinder is inscribed inside as shown in the figure. The cylinder inside has a radius r and a height of h, while the outside cone has a radius R and a height of H. The volume of the cylinder must be equal to 0.1 m^3.

(i) We wish to minimize the volume of the cone which is given by $V_{cone} = \frac{1}{3}\pi R^2 H$. Find the optimal value of R as a function of r at this optimum. (Note that R must be $> r$)

(ii) Calculate the minimum volume of the cone if $r = 0.1$ m.

(iii) Prove that the calculated volume is a minimum by showing that when $R = R_{opt} + \Delta$ and when $R = R_{opt} - \Delta$, the volumes of the cones at these two radii are larger than the calculated minimum volume.



Formulas:

Volume (cone) $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi R^2 H$, Volume (sphere) $= \frac{4}{3}\pi R^3$, Volume (cylinder) $= \pi r^2 h$

20. *Formulating a Compressor Storage Problem*

Acme Inc. engages in the business of building large and expensive compressors. The company operates 3 plants, called Plant A, B, and C, in different locations which have different manufacturing capacities (number of compressors built at the end of each production week). Because of their sizes, all the compressors built at each plant must be transported to a warehouse for storage at the end of each production week. The company owns three warehouses in different locations, which also have different storage capacities. The total cost associated with storing one compressor (e.g. transportation cost, space rental fee, etc.) at each warehouse is known and given in the table below.

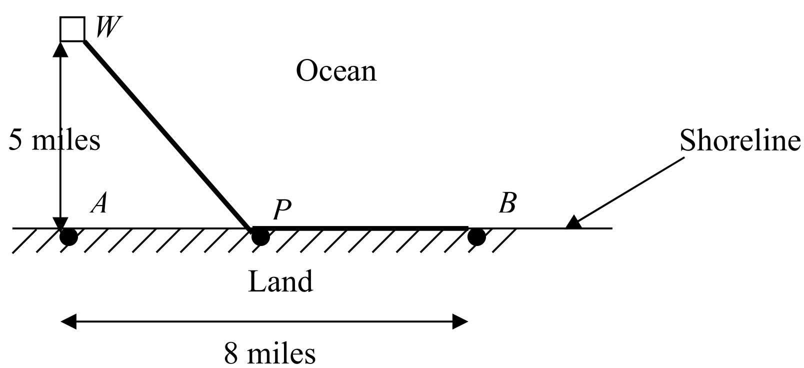
As a chemical engineer working for Acme, you are asked to formulate this optimization problem and classify it. The objective is to minimize the total cost of storing all compressors manufactured at the end of a production week at the 3 warehouses.

Cost of Storing a Compressor from a Plant at a Warehouse

	Warehouse 1	**Warehouse 2**	**Warehouse 3**	**Plant Capacity**
Plant *A*	15	20	50	30
Plant *B*	15	30	40	25
Plant *C*	30	10	10	40
Max. Storage Capacity	40	30	35	

21. *Minimizing the Piping Cost for an Offshore Oil Well*

An offshore oil well is located in the ocean at a point *W*, which is 5 miles from the closest shorepoint *A* on a straight shoreline (see the figure below). The oil is to be piped to a shorepoint *B* that is 8 miles from *A* by piping it on a straight line under water from *W* to some shorepoint *P* between *A* and *B* and then on to *B* via a pipe along the shoreline. If the cost of laying pipe is \$100,000 per mile under water and \$75,000 per mile over land, formulate this piping problem as an optimization problem to minimize the cost of laying the pipe, classify it as either unconstrained NLP, constrained NLP, LP, MILP, or MINLP, and determine analytically where the point *P* should be located that gives the least possible cost? Compute this minimum cost as well.



22. *Characterization of Stationary Points*

Identify and characterize (as local/global minimum, local/global maximum, or saddle point) the stationary points of the following functions. Compute the value of the function at each stationary point as well.

(a) $f(x) = x^4 + x^3 - 2x^2 - 2\exp(-x^2)$

(b) $f(x) = (x-1)^2 \exp[x(x+3)]$

(c) $f(x_1, x_2) = \ln(x_1 x_2) + x_1 x_2 - x_2^2 - 2x_1$

(d) $f(x_1, x_2) = 2x_1^3 + 4x_1 x_2^2 - 10x_1 x_2 + x_2^2$

(e) $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_3^2 - x_2^2 x_3 + \ln(x_3) - 4x_3 + 4$

(f) $f(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 + x_2^2$

(g) $f(x_1, x_2) = x_1 x_2 - 4\sin(x_1) + \ln\left(\frac{x_1}{x_2}\right) \qquad 0 \le x_1, x_2 \le \pi$

(h) $f(x_1, x_2, x_3) = \exp(-x_1^2) + 2x_1^2 + x_2^2 + x_2 x_3 + x_3^3$

(i) $f(x_1, x_2) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6$

(j) $f(x_1, x_2, x_3) = \frac{x_1}{(1-x_2)} - \exp(x_2) + 2x_1^3 - 4x_1 + 2x_2 + x_1 x_2 x_3$

(k) $f(x, y) = x^2 y + xy^2 - 2x^2 - 2y^2$

(l) $f(x, y) = (x^2 + y^2) \exp(y^2 - x^2)$

(m) $f(x_1, x_2) = (x_1 - 2)\exp(x_1 + x_2^2) + x_2$

(n) $f(x_1, x_2) = 2x_1^2 + x_1^2 x_2 - x_1 x_2^2 + 2x_1 x_2 + 8x_1$

23. *Solving a Heat Transfer Minimization Problem*

Refer to Problem 10, which is a problem to minimize the total annual cost of a heat exchanger. After transforming this constrained problem into an unconstrained one,

(a) Use the function *fminunc* in MATLAB to solve for the minimum annual cost.

(b) Use the function *fminsearch* in MATLAB to solve for the minimum annual cost.

24. *Optimization of a Solvent-Extraction Operation*

A solvent-extraction operation is carried out continuously in a plate column with gravity flow. The unit is operated 24 hr/day. A feed rate of 1500 ft^3/day must be handled 300 days per year. The allowable velocity per square foot of cross-sectional tower area is 40 ft^3 of combined solvent and charge per hour. The annual fixed costs for the installation can be predicted from the following equation:

$$C_F = 6{,}800F^2_{SF} - 48{,}000F_{SF} + 120{,}000 \quad \$/\text{year}$$

where F_{SF} = cubic feet of solvent per cubic foot of feed. Operating and other variable costs depend on the amount of solvent that must be recovered, and those costs are \$0.04 for each cubic foot of solvent passing through the tower.

(a) What tower diameter should be used for optimum conditions of minimum total cost per year?

(b) What is the answer in Part (a) if the annual fixed costs for the installation follow the equation below:

$$C_F = 400F^4_{SF} - 1{,}800F^3_{SF} - 12{,}000F_{SF} + 120{,}000 \quad \$/\text{year}$$

Write a MATLAB program to implement the following methods to find the optimum diameter D_{opt}, correct to 3 decimal places, i.e. $| D_{opt,k+1} | - | D_{opt,k} | \geq 0.001$.

(i) Newton-Raphson method
(ii) Quasi-Newton method using a step size of $h = 0.1$
(iii) Secant method

25. *Single-Variable Search Using 3 Different Techniques*

Carry out a single-variable search of the function

$$f(x) = x + 4/x \quad \text{on the interval } [1, 5]$$

using (a) Newton-Raphson method
(b) Quasi-Newton method using a step size of $h = 0.1$
(c) Secant method

Carry out 5 iterations in each method using Excel. Also determine the true minimum of $f(x)$ analytically and compare the accuracy of the minima estimated by these three methods.

26. *Solving for the Optimal Height and Diameter of an Absorption Tower*

Refer to Problem 4 again. Recall that G_s, the superficial gas velocity, is the decision variable to be minimized in the cost function. Given that the optimal value of G_s lies between 1000 and 2000 lbm/ft^2-hr, do the following:

(a) Use *fminbnd* function in MATLAB to solve for the optimal tower height, diameter, and minimum annual cost. What is the number of iterations, the number of function evaluations, and the algorithm used as reported by MATLAB?

(b) Write a MATLAB script file to implement Golden Section search to find the optimal G_s, accurate to 1 decimal place. What is the total number of iterations required?

27. *Solving for a Maximum Using Golden Section*

Find the maximum of the function xe^{-x} over the interval $0 \le x \le 4$ by performing 6 function evaluations with the Golden Section search by hand. What is the accuracy of the result after 6 iterations? Confirm analytically that the solution is in fact a local maximum. Prove that it is a global maximum.

28. *Steepest Descent with Golden Section*

Consider a chemical process for which there are 2 design variables x and y. After cost estimation and profitability analysis, it is determined that it is desirable to find the values of x and y that minimizes the cost function C:

$$C = 6.5x^{0.5} + \frac{3000}{xy} + 5y + 2x^2 + 60$$

The initial guess is $x^{(0)} = 2$ and $y^{(0)} = 7$.

In this problem you are asked to perform one iteration of the steepest descent method for solving this problem. That is, find $x^{(1)}$ and $y^{(1)}$, the next estimates of the optimum.

For the variable optimization with respect to the parameter λ (line search, which indicates how far to move in the steepest descent direction), use the Golden Section method. Assume C is unimodal with respect to λ, and that $0 \le \lambda \le 0.1$. You need to perform only 3 iterations of the Golden Section method (so you will have to evaluate C only 4 times). You must show all intermediate results in each iteration of Golden Section. Do not use Excel, MATLAB, or the programmable feature in your calculators and do not show your answers in the form of tables.

You should express your answer in terms of an interval. That is, determine what two numbers $x^{(1)}$ lies between and what two numbers $y^{(1)}$ lies between. Carry 4 decimal places in all your calculations.

29. *Unconstrained Optimization of a Two-Variable Problem*

Consider the following unconstrained minimization problem of two variables.

$$\text{Minimize } f(x_1, x_2) = 2x_1^2 + x_1^2 x_2 - x_1 x_2^2 + 2x_1 x_2 + 8x_1$$

Carry out one iteration of Steepest Descent Search and one iteration of Newton's method, starting at $\boldsymbol{x}^{(0)} = [1.0\ 1.0]^T$ in both cases. Carry 4 decimal places in all your calculations.

Note that you must perform a line search in Steepest Descent Search using calculus. Compute the objective value after one iteration in both algorithms.

30. ***Line Search with Golden Section in a 2-Variable Problem***

A one-dimensional search is to be carried out from the point [0, 3] using $\boldsymbol{s} = [1\ 1]^T$; $\boldsymbol{s}$ is the search direction on the function:

$$f(x) = (x_1 - 1)^2 + 4(x_2 - 2)^2$$

Use Golden Section method to find the minimum along this search direction (with respect to λ), with an accuracy of 4 decimal places. Also, find this optimum value analytically and show that it is (–0.6, 2.4).

31. ***Line Search with Golden Section in a 2-Variable Problem***

A one-dimensional search is to be carried out from the point [1, 3] using $\boldsymbol{s} = [1\ {-2}]^T$; $\boldsymbol{s}$ is the search direction on the function:

$$f(\boldsymbol{x}) = 2x_1^3 + 4x_1x_2^2 - 10x_1x_2 + x_2^2$$

Implement Golden Section method in MATLAB to find the minimum along this search direction (with respect to λ), with an accuracy of 4 decimal places. Also, use *fminbnd* to find the optimum value of the step size λ.

32. ***Conjugate Search Directions for a Quadratic Function***

The general equation in matrix form for a quadratic function can be expressed as:

$$f(\boldsymbol{x}) = a + b^T\boldsymbol{x} + \tfrac{1}{2}\,\boldsymbol{x}^T C\boldsymbol{x}$$

where b^T is a ***1xn*** matrix and C is an ***nxn*** symmetric matrix, and n is the total number of variables.

(a) For the function

$$f(\boldsymbol{x}) = 5x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_2 - x_1x_3 - 4x_2 + 5x_3 + 8$$

show that the vectors $[1\ \ 0\ \ 0]^T$ and $[1\ \ 1\ \ 12]^T$ are conjugate directions with respect to the C matrix.

(b) Compare the C matrix with the Hessian matrix $H(\boldsymbol{x})$ of $f(\boldsymbol{x})$ and comment on their relationship.

(c) Is the function in Part (a) strictly convex, convex, strictly concave, concave, or none of the above?

33. *Combined Search Techniques in a Multivariable Minimization Problem*

Consider the following unconstrained minimization problem of two variables.

$$\text{Minimize} \quad f(x_1, x_2) = x_1^2 - x_1 x_2 + \ln(x_1 + x_2)$$

Carry out two iterations of Conjugate Direction Search, starting at $\boldsymbol{x}^{(0)} = [1.0\ 1.0]^T$. In the first iteration, use the Steepest Descent as the search direction. In the second iteration, your search direction must be made conjugate to the search direction in the first iteration. Carry 4 decimal places in all your calculations. Note that you must perform a line search in both iterations using calculus. Compute the objective value of each iteration.

34. *Unconstrained Optimization of a Two-Variable Problem*

Consider the following unconstrained optimization problem of two variables. Carry out one iteration of each method below to minimize the given objective function.

$$\text{Minimize} \quad f(x) = x_1^4 + 2x_1x_2^2 - x_1x_2$$

Method 1: Starting at $[x_1, x_2]^T = [1, 0]^T$, carry out one iteration of Steepest Descent search. Note that you must minimize λ, whose optimal value is less than 0.2, in the line search accurate to 4 decimal places. You may choose any line-search (one-variable) optimization technique but a fast convergence method is recommended. You must show all work instead of using your programmable calculator to give the final answer.

Method 2: Starting at $[x_1, x_2]^T = [1, 0]^T$, carry out one iteration of Conjugate Direction search. This search direction must be conjugate to the steepest descent search direction in Method 1. Fix the value of the second element of your conjugate vector to 1 and calculate its first element, i.e. $\boldsymbol{s} = [s_1, 1]^T$. Once again, you must minimize λ, whose value is less than 0.2, in the line search accurate to 4 decimal places. You may choose any line-search (one-variable) optimization technique but a fast convergence method is recommended. You must show all work instead of using your programmable calculator to give the final answer.

Method 3: Starting at $[x_1, x_2]^T = [1, 0]^T$, carry out one iteration of Newton's method

Calculate $f(\boldsymbol{x})$ in each method after one iteration. What are the absolute relative percentage deviations of x_1, x_2, and $f(\boldsymbol{x})$ of the three methods when compared to the global minimum in this problem? Carry 4 decimal places in all your calculations.

<u>Formula</u>: The inverse of a square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is* $\dfrac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

where $\det(A) = ad - bc$

35. *Using Steepest Descent to Solve a Multivariable Minimization Problem*

Consider an unconstrained optimization problem with two decision variables x and y. It is desirable to find the values of x and y that <u>minimizes</u> the objective function $f(x, y)$:

$$f(x,y) = (y - x^2)^2 + (5 - x)^2$$

Carry out one iteration of the Steepest Descent method. Use an initial guess of $x = 2$ and $y = 5$. In your line search, carry out 5 iterations of Golden-Section Search.

36. *Conjugate-Gradient and Newton's Methods for a Multivariable Problem*

For the following quadratic function,

$$f(\boldsymbol{x}) = 4(x_1 - 5)^2 + (x_2 - 6)^2$$

starting at $\boldsymbol{x}^0 = [1 \;\; 1]^T$.

(a) Use the Fletcher-Reeves (conjugate-gradient) search to find the minimum.

(b) Use Newton's method to find the minimum.

37. *Newton's Methods for Another Multivariable Problem*

Solve the following problem by Newton's method:

$$\text{Minimize} \quad f(\boldsymbol{x}) = 1 + x_1 + x_2 + x_3 + x_4 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 + x_1^2 + x_2^2 + x_3^2 + x_4^2$$

starting from $\boldsymbol{x}^0 = [-3 \;\; -30 \;\; -4 \;\; -0.1]^T$.

38. *Steepest Descent and Conjugate Direction Methods*

Consider the maximization of

$$f(\boldsymbol{x}) = 12x_1 - 2x_1^2 - 2x_2^2 + 2x_1x_2 - 12$$

(a) Find the stationary point analytically, and prove that it is a local maximum.

(b) Carry out 3 iterations of steepest descent (with the line search) from the point $[1 \;\; 1]^T$. What are the values of x_1, x_2, and $f(\boldsymbol{x})$ after 3 iterations?

(c) Determine the maximum using conjugate search directions. Since the number of variables is 2 and $f(\boldsymbol{x})$ is quadratic, it should take you only 2 iterations to find the maximum.

39. *Conjugate-Gradient Method, Newton's Method, and fminunc for a Multivariable Problem*

For the following optimization problem,

$$\text{Minimize } f(\boldsymbol{x}) = \exp(-x_1^2) + 2x_1^2 + x_2^2 + x_2x_3 + x_3^3$$

Write a MATLAB program to implement the following:

(a) 10 iterations of the Fletcher-Reeves (conjugate-gradient) search to find the minimum starting at $\boldsymbol{x}^0 = [0\ -0.5\ \ 0.5]^T$. Do not perform the line search in each iteration. Instead, use a fixed step size of $\lambda = 0.1$.

(b) 5 iterations of Newton's method to find the minimum starting at $\boldsymbol{x}^0 = [0\ \ -1\ \ 1]^T$.

(c) Use *fminunc* function to solve for the minimum at $\boldsymbol{x}^0 = [1\ \ 1\ \ 1]^T$. Do not use the default algorithm of Quasi-Newton line search because it will not converge the problem at the given starting point. Instead, use the trust-region Newton algorithm.

40. *Sequential Simplex Method and MATLAB's fminsearch*

For the following function

$$f(\boldsymbol{x}) = [\exp(x_1)][4x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2 + 1]$$

(a) Carry out 5 steps of sequential simplex method to find the minimum, using a simplex with a length of 0.2 and a starting point of $\boldsymbol{x}^0 = [2\ -2]^T$.

(b) Use *fminsearch* in MATLAB to find the minimum starting at $\boldsymbol{x}^0 = [2\ -2]^T$.

41. *Combined Search Techniques in a Multivariable Minimization Problem*

Consider a chemical process for which there are 2 design variables x_1 and x_2. After cost estimation and profitability analysis, it is determined that it is desirable to find the values of x_1 and x_2 that <u>minimizes</u> the cost function $f(\boldsymbol{x})$:

$$f(x_1, x_2) = x_1^2 + 2x_1x_2^2 - x_1x_2$$

In searching for the minimum, perform 3 iterations using the following combination of search techniques:

Iteration 1: use Steepest Descent search
Iteration 2: use a search direction which is conjugate to the search direction in Iteration 1
Iteration 3: use Newton's method

Use an initial guess of $x_1 = 2$ and $x_2 = 1$. In Iteration 1 and 2, you must carry out a line search (i.e. minimize the step size λ completely), and in each iteration use the solution from the previous iteration as the starting point. Calculate and report the value of $f(\boldsymbol{x})$ in each iteration as well.

42. *Profit Maximization in a Chemical Process as an LP*

(a) A chemical company makes two products, Product A and Product B, which sell for a profit of $3,000 per unit and $3,200 per unit, respectively. It takes 32 minutes to produce Product A and 40 minutes to produce Product B. The total production time of each day is 660 minutes. To stay in business, the company must produce at least 5 units of Product B but no more than 14 units per day. At the same time, the market demand dictates that the company should not produce more than a combined total of 20 units of Product A and Product B. Finally, the machinery used in producing Product A and Product B requires that both products must be produced in multiples of 5 units. Formulate this problem as an ILP (because the number of product units must be a whole number) and solve for the maximum profit per day by graphical method.

(b) A chemical company produces 2 kinds of product, namely Saccharin and Nutrasweet which are sugar substitutes because of their extreme sweetness. Saccharin generates a profit of $4100 per batch, requires 4 hours of production per batch in Plant 1 and 2 hours per batch in Plant 2. Nutrasweet has a profit contribution of $5900 per batch, requires 2 hours per batch in Plant 1 and 3 hours per batch in Plant 2 to produce. The available time in one production cycle in Plant 1 and Plant 2 are 160 hours and 180 hours, respectively. Each hour used in Plant 1 costs $90, while each hour spent in Plant 2 costs $60. Formulate the problem as an LP in order to maximize the profit in one production cycle. Solve the LP problem graphically. Solve the same problem again using the Simplex algorithm.

43. *Formulation of an LP Problem*

Anti-H5N1 Inc. is a pharmaceutical company specialized in producing an expensive vaccine to inoculate humans against the avian flu. The vaccines come in two varieties, one for immunization against Strain A (called AFVAX-A) and one against Strain B (called AFVAX-B). A facility is used to produce both AFVAX-A and AFVAX-B. Because of the high demands for its products, the company must produce a total of at least 14 batches in one production cycle, of which at least 3 batches must be AFVAX-A and at least 2 batches must be AFVAX-B. It takes 8.0 hours to produce one batch of the Strain-A vaccine and 6.25 hours to produce one batch of the Strain-B vaccine. A maximum of 100 operation hours is available at the facility in one production cycle. The net profit for the Strain-A vaccine is $25,000 per batch, while that of the Strain-B is $19,800 per batch.

Anti-H5N1 Inc. wants to know how much of AFVAX-A and AFVAX-B they must produce in one production cycle in order to achieve the highest possible profit.

(a) Formulate an LP model for this problem. Treat all variables as continuous and use the graphical method to solve this model.

(b) Now treat the model as an ILP problem, i.e. the number of batches must be an integer. Use the graphical method again to solve for the optimum. You must show all your work to get full credit.

44. *Optimal Manufacturing of Candies*

A confectioner manufactures 2 kinds of candy bars: Ergies (packed with energy for kiddies) and Nergies (the "low-calorie" nugget for weight watchers without will power). Ergies sell at a profit of 50 cents per box while Nergies have a profit of 60 cents per box. The candy is processed in 3 main operations: blending, cooking, and packaging. The following table records the average time in *minutes* required by each box of candy, for each of the three activities.

	Blending	**Cooking**	**Packaging**
Ergies	1	5	3
Nergies	2	4	1

During each production run, the blending equipment is available for a maximum of 14 machine hours, the cooking equipment for at most 40 machine hours, and the packaging equipment for at most 15 machine hours.

(a) If each machine can be allocated to the making of either type of candy at all times that it is available for production, determine how many boxes of each kind of candy the confectioner should make in order to realize the maximum profit. Formulate this optimization problem as a linear program and use a graphical technique to solve for the optimum. Also, compute the maximum profit in dollars (Note: 100 cents = 1 US dollar).

(b) The marketing department proposed that the confectioner add a third kind of candy bar named Munchies into the production line in order to try to make more profit. Munchies has a profit of 80 cents per box with the following operation times: blending time = 2 minutes, cooking time = 5 minutes, packing time = 2 minutes. Due to some adjustments in the production equipment required to accommodate the making of Munchies, the maximum hours available for the blending equipment is now reduced by 3 machine hours, the maximum for the cooking equipment is reduced by 2 machine hours, and the maximum for the packaging equipment is reduced by 2 machine hours.

As a ChEPS graduate working for the confectioner, do you agree with this proposal from marketing? Support the conclusion you make analytically (if you use linear programming to prove your answer, use the Simplex algorithm to solve the problem).

There is no minimum number of boxes the confectioner is obligated to make for each kind of candy bars.

45. *Optimal Production of Two Chemicals as an LP Problem*

A chemical manufacturer plans to make two products P and Q by batch reaction. By varying reaction conditions and batch time, both products can be made from the same reactants A and B as follows:

Reaction 1: $0.5 \text{ kg } A + 0.6 \text{ kg } B \rightarrow 1 \text{ kg } P + \text{waste}$

Required batch time (including time to charge and empty reactor) = 2.5 hours

Reaction 2: $0.2 \text{ kg } A + 0.9 \text{ kg } B \rightarrow 1 \text{ kg } Q + \text{waste}$

Required batch time (including time to charge and empty reactor) = 5 hours

P can be sold at a profit of \$ 55/ton, and Q at a profit of \$ 80/ton. The company plans to use the same batch reactor to produce both products. So part of the time the reactor will be used to produce P using reaction 1, and the remaining time the reactor will be used to produce Q using reaction 2. In either case, each batch produces 0.5 ton of product P or Q. The reactor operates 330 days/year for 24 hours/day. The company is able to purchase up to 600 tons/year of A and up to 1000 tons/year of B. Additional supplies of A and B are not available.

(a) Assuming you want to maximize the total annual profit from selling P and Q, formulate this problem as a linear programming (LP) problem.

(b) How much P should you produce, and how much Q should you produce? Use the Simplex algorithm to solve for the optimum.

46. *Formulation of an LP Problem*

A Thai shrimp company with a farming area of 10 rais wishes to optimize the growing of 3 kinds of shrimps, namely tiger shrimps, prawns, and rock shrimps. The company can sell tiger shrimps at 100 baht a kilogram, prawns at 160 baht a kilogram, and rock shrimps at 200 baht a kilogram. The average yield per rai is 1,500 kilograms of tiger shrimps, 1,000 kilograms of prawns, and 800 kilograms of rock shrimps. Labor required per rai during each harvest cycle is 3 man-days, 6 man-days, and 4 man-days for tiger shrimps, prawns, and rock shrimps, respectively. No more than 50 man-days are available during the harvest cycle, and each worker is paid 300 baht per man-day. Feed and other operating expenses cost 60 baht per kilogram, while feed requirements are: 300 kilograms per rai of tiger shrimps, 450 kilograms per rai of prawns, and 500 kilograms per rai of rock shrimps.

(a) Based on this information, formulate this problem as an LP model to determine the optimal usage of the farming area such that the profit in each harvest cycle is maximized.

(b) Carry out 1 iteration of Simplex to solve this LP model. Is the solution optimal after 1 iteration?

47. *Simplex Algorithm, I*

Consider the following LP minimization problem:

Minimize $\quad f(\mathbf{x}) = -2x_1 + x_2 - x_3$

s.t.

$$
\begin{aligned}
x_1 + x_2 &\leq 12 \qquad (1)\\
-x_1 + 4x_2 - 2x_3 &\geq -6 \qquad (2)\\
2x_1 + x_2 - 2x_3 &= 6 \qquad (3)\\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

Solve the LP problem and find the minimum by starting Simplex with the feasible basic solution in which $x_2 = x_3 = 0$. You are not allowed to eliminate any variables and reduce the number of constraints when using Simplex.

48. *Simplex Algorithm, II*

Use the Simplex algorithm to solve for the optimum of the following LP:

Minimize $\quad f(\mathbf{x}) = x_1 + 2x_2 + 2x_3 - x_4$

s.t.

$$
\begin{aligned}
x_1 + 2x_2 - 2x_3 &\geq 0 \qquad (1)\\
2x_1 - 4x_2 - x_3 + x_4 &\leq 14 \qquad (2)\\
-2x_1 + x_2 + 3x_3 - x_4 &\geq -6 \qquad (3)\\
x_1 + x_2 + x_3 + x_4 &= 12 \qquad (4)
\end{aligned}
$$

$$x_1, x_2, x_3, x_4 \geq 0$$

You are not allowed to eliminate any variables and thus reduce the number of constraints when using Simplex. A close inspection of the LP formulation shows that one feasible basic solution can be found by making x_2 and the slack/surplus variables of Constraint (1) and Constraint (3) non-basic variables. Start your Simplex with this basic solution.

49. *Simplex Algorithm, III*

Use the Simplex algorithm to solve for the optimum of the following LP:

Maximize $\quad f(\mathbf{x}) = x_1 - 2x_2 + 4x_3$

s.t.

$$2x_1 + x_2 - x_3 \geq 6$$

$$x_1 + x_2 + x_3 \leq 20$$
$$x_2 + x_3 \geq 8$$

$$x_1 \geq 0,\ x_2 \text{ is unrestricted},\ x_3 \geq 0$$

You are <u>not</u> allowed to eliminate any constraint when using Simplex.

50. *Steel Blending Problem*

The Pittsburgh Steel (PS) Co. has been contracted to produce a new type of steel which has the following tight quality requirements:

Content	At Least	Not More Than
Carbon	3.00%	3.50%
Chrome	0.30%	0.45%
Manganese	1.35%	1.65%
Silicon	2.70%	3.00%

PS has the following materials available for mixing up a batch:

	Cost/lb.	Percent Carbon	Percent Chrome	Percent Manganese	Percent Silicon	Amount Available
Pig Iron 1	0.03	4.0	0	0.9	2.25	Unlimited
Pig Iron 2	0.0645	0	10.0	4.5	15.0	Unlimited
Ferro-Silicon 1	0.065	0	0	0	45.0	Unlimited
Ferro-Silicon 2	0.061	0	0	0	42.0	Unlimited
Alloy 1	0.10	0	0	60.0	18.0	Unlimited
Alloy 2	0.13	0	20.0	9.0	30.0	Unlimited
Alloy 3	0.119	0	8.0	33.0	25.0	Unlimited
Carbide	0.08	15.0	0	0	30.0	Unlimited
Steel 1	0.021	0.4	0	0.9	0	200 lb.
Steel 2	0.02	0.1		0	0.3	0 200 lb.
Steel 3	0.0195	0.1	0	0.3	0	200 lb.

A one-ton (2000 lb.) batch must be blended which satisfies the quality requirements stated earlier. The problem now is what amounts of each of the eleven materials should be blended together so as to minimize the cost but satisfy the quality requirements. An experienced steel man claims that the least cost mix will not use more than nine of the eleven raw materials. Formulate this problem as an LP problem. To verify the steel man's claim, use LINDO to solve for the optimum. Is the steel man's claim correct or wrong?

51. *Reclaiming Solid Wastes*

The SAVE-IT COMPANY operates a reclamation center that collects four types of solid waste materials and treats them so that they can be amalgamated into a salable product (Treating and amalgamating are separate processes.) Three different grades of this product can be made (see the first column of Table 1), depending upon the mix of the materials used. Although there is some flexibility in the mix for each grade, quality standards may specify the minimum or maximum amount allowed for the proportion of a material in the product grade. (This proportion is the weight of the material expressed as a percentage of the total weight for the product grade.) For each of the two higher grades, a fixed percentage is specified for one of the materials. These specifications are given in Table 1 along with the cost of amalgamation and the selling price for each grade.

The reclamation center collects its solid waste materials from regular sources and so is normally able to maintain a steady rate for treating them. Table 2 gives the quantities available for collection and treatment each week, as well as the cost of treatment, for each type of material.

The Save-It Co. is solely owned by Green Earth, an organization devoted to dealing with environmental issues, so Save-It's profits are used to help support Green Earth's activities. Green Earth has raised contributions and grants, amounting to $30,000 per week, to be used exclusively to cover the entire treatment cost for the solid waste materials. The board of directors of Green Earth has instructed the management of Save-It to divide this money among the materials in such a way that *at least half* of the amount available of each material is actually collected and treated. These additional restrictions are listed in Table 2.

Within the restrictions specified in Tables 1 and 2, management wants to determine the *amount* of each product grade to produce *and* the exact *mix* of materials to be used for each grade. The objective is to maximize the net weekly profit (total sales income *minus* total amalgamation cost), exclusive of the fixed treatment cost of $30,000 per week that is being covered by gifts and grants. Formulate this problem as an LP and solve it using LINDO.

Table 1: Product data for Save-It Co.

Grade	**Specification**	**Amalgamation**	**Selling Price**
	Cost per Pound ($)	**per Pound ($)**	
A	Material 1: Not more than 30% of total Material 2: Not less than 40% of total Material 3: Not more than 50% of total Material 4: Exactly 20% of total	3.00	8.50
B	Material 1: Not more than 50% of total Material 2: Not less than 10% of total	2.50	7.00

	Material 4: Exactly 10% of total		
C	Material 1: Not more than 70% of total	2.00	5.50

Table 2: Solid waste materials data for Save-It Co.

Material	Pounds per Week Available	Treatment Cost per Pound ($)	Additional Restrictions
1	3,000	3.00	1. For each material, at least half of the pounds per week available should be collected and treated.
2	2,000	6.00	
3	4,000	4.00	
4	1,000	5.00	2. $30,000 per week should be used to treat these materials.

52. ***The Method of Lagrange Multipliers, I***

Use the method of Lagrange Multiplier to solve for all stationary points of the following constrained optimization problem:

$$f(x) = x_1^2 x_2 - x_1 x_2 + 3x_2$$

$$s.t. \quad x_1 x_2 \le 10$$

$$x_1 + x_2 = 4$$

You need not characterize the stationary points, but do compute the objective function value of each stationary point and their Langrage multipliers.

53. ***The Method of Lagrange Multipliers, II***

Consider the following nonlinear constrained optimization problem:

$$\text{Minimize} \quad f(\mathbf{x}) = x_1^2 + 2x_1x_2 + (x_2 - 1)^2$$

$$\text{subject to} \quad g(\mathbf{x}) = x_1^2 + x_2^2 - 9 \le 0$$

$$h(\mathbf{x}) = x_1 + 2x_2^2 = 10$$

Use the Method of Lagrange Multipliers to find all the stationary points. For each stationary point you find, be sure to calculate the values of the Lagrange Multipliers as well as the values of $f(\mathbf{x})$. You do not need to classify each stationary point.

54. *The Method of Lagrange Multipliers, III*

Use the method of Lagrange Multiplier to solve for <u>all</u> stationary points of the following constrained optimization problem:

$$f(x) = x_1 x_2$$

$$s.t. \quad x_1 + 2x_2 \le 2$$

$$2x_1^2 - x_2 = 6$$

You need <u>not</u> characterize the stationary points, but do compute the objective function value of each stationary point and their Langrage multipliers.

55. *Method of Lagrange Multipliers and fmincon, I*

(a) Find all the stationary points of the following constrained NLP by the method of Lagrange Multiplier. You must calculate the values of all variables in the Lagrangian L, including the values of the objective function and the multipliers. But do not classify each stationary point.

$$f(x_1, x_2) = x_1 x_2^2$$

$$\text{s.t.} \quad x_1 + 2x_2 = 10$$

$$x_1^2 + x_2^2 \le 25$$

(b) Write down the MATLAB syntax (input commands and M-files) and use *fmincon* to solve the following constrained NLP problem:

$$\text{Maximize } f(\mathbf{x}) = 3x_1 \exp(-0.1x_1x_6) + 4x_2 + x_3^2 + 7x_4 + \frac{10}{x_5} + x_6$$

s.t.

$x_2 x_6 = 5$ (Constraint 1)

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$ (Constraint 2)

$2x_1 + x_2 + x_3 + 3x_4 \ge 2$ (Constraint 3)

$\frac{x_1}{x_2} + x_3^2 x_4^3 = 1$ (Constraint 4)

$-8x_1 - 3x_2 - 4x_3 + x_4 - x_5 + x_6 \ge -10$ (Constraint 5)

$-2x_1 - 6x_2 - x_3 - 3x_4 - x_6 \ge -13$ (Constraint 6)

$-x_1 - 4x_2 - 5x_3 - 2x_4 \ge -18$ (Constraint 7)

$\sqrt{x_5} + x_6 \le 6$ (Constraint 8)

$-20 \le x_i \le 20 \quad i = 1, \ldots, 6$

Use an initial guess of $x_i = 1.0$, $i = 1, \ldots, 6$. The solution from MATLAB is $x_1 = 1.4158$, $x_2 = 1.5583$, $x_3 = -3.0346$, $x_4 = 0.2149$, $x_5 = 6.6369$, and $x_6 = 3.2087$ with $f(\boldsymbol{x}) = 24.3585$. Which inequality constraint(s) is/are active (identify by constraint number)? Be sure to write neatly and pay attention to details such as semicolons, periods, etc., since the syntax must be precise.

56. *Method of Lagrange Multipliers and fmincon, II*

(a) Use the method of Lagrange multipliers to determine the stationary points of the following nonlinear constrained problem. There is no need to classify the calculated stationary points as maximum, minimum, or saddle point, but you must calculate the values of the Lagrange multiplier and the objective function of each stationary point.

Minimize $f(x, y) = xy^2$

s.t. $x^2 - y^2 \le 16$

$x - y = 3$

(b) Write down the MATLAB syntax (input commands and M-files) and use *fmincon* to solve the following constrained NLP problem:

Minimize $f(\boldsymbol{x}) = x_2 \exp(x_1 x_3) - x_3 x_4 + x_4^3 + \dfrac{2}{x_3^2} - \ln(x_1 + x_4)$

s.t.

$$4x_1 - x_2^2 + x_3 + x_4 = -11 \qquad \text{(Constraint 1)}$$

$$2\sqrt{x_2} - \frac{x_1}{x_4} + \exp(-x_3) \ge 30 \qquad \text{(Constraint 2)}$$

$$4x_1 - 2x_2 + 2x_3 + x_4 \le 0 \qquad \text{(Constraint 3)}$$

$$4x_1 + 2x_2 + 4x_3 - x_4 = -4 \qquad \text{(Constraint 4)}$$

$$6x_1 + 4x_2 - 5x_3 + 2x_4 \ge 50 \qquad \text{(Constraint 5)}$$

$$2x_1 + \exp\left(\frac{x_2 x_3}{x_4}\right) \le 20 \qquad \text{(Constraint 6)}$$

$$2x_1 + x_2 + x_3 + x_4 = 14 \qquad \text{(Constraint 7)}$$

$$-x_1 x_3 + \frac{8x_2}{x_4} = 15 \qquad \text{(Constraint 8)}$$

$$-100 \le x_i \le 100 \qquad i = 1, \ldots, 4$$

Use an initial guess of $x_i = 1$, $i = 1, \ldots, 4$. Be sure to write neatly and pay attention to details such as semicolons, periods, etc., since the syntax must be precise.

Is this optimization under-determined, uniquely determined, or over-determined? The solution from MATLAB is $x_1 = 2.5$, $x_2 = 5.0$, $x_3 = -4.0$, and $x_4 = 8.0$ with $f(\boldsymbol{x}) = 541.7739$. Which inequality constraint(s) is/are active (identify by constraint number)?

57. *Using fmincon to Solve 3 NLPs*

(a) Westerberg and Shah (1978) in Swaney (1990) studied the following NLP problem:

$$
\begin{aligned}
\text{Min} \quad & 35x_1^{0.6} + 35x_2^{0.6} \\
\text{s.t.} \quad & 600x_1 - 50x_3 - x_1x_3 + 5000 = 0 \\
& 600x_2 + 50x_3 - 15000 = 0 \\
& (0, 0, 100) \le \boldsymbol{x} \le (34, 17, 300)
\end{aligned}
$$

in which one local optimum ($f(\boldsymbol{x}) = 286.943$) and the global optimum ($f(\boldsymbol{x}) = 189.311627$) were reported by Ryoo and Sahinidis (1995). Use *fmincon* to find both solutions.

(b) Stoecker (1971) in Liebman *et al.* (1986) studied the following insulated steel tank design NLP problem:

$$
\begin{aligned}
\text{Min} \quad & 400x_1^{0.9} + 1000 + 22(x_2 - 14.7)^{1.2} + x_4 \\
\text{s.t.} \quad & x_2 = \exp[-3950/(x_3 + 460) + 11.86] \\
& 144(80 - x_3) = x_1x_4 \\
& (0, 14.7, -459.67, 0) \le \boldsymbol{x} \le (15.1, 94.2, 80, \infty)
\end{aligned}
$$

in which the global optimum ($f(\boldsymbol{x}) = 5194.866243$) was reported by Ryoo and Sahinidis (1995). Use *fmincon* to find the global optimum.

(c) Stephanopoulos and Westerberg (1975) studied the following NLP problem of a design of three-stage process system with recycle.

$$
\begin{aligned}
\text{Min} \quad & x_1^{0.6} + x_2^{0.6} + x_3^{0.4} - 4x_3 + 2x_4 + 5x_5 - x_6 \\
\text{s.t.} \quad & -3x_1 + x_2 - 3x_4 = 0 \\
& -2x_2 + x_3 - 2x_5 = 0 \\
& 4x_4 - x_6 = 0 \\
& x_1 + 2x_4 \le 4 \\
& x_2 + x_5 \le 4 \\
& x_3 + x_6 \le 6 \\
& (0, 0, 0, 0, 0, 0) \le \boldsymbol{x} \le (3, 4, 4, 2, 2, 6)
\end{aligned}
$$

in which the global optimum ($f(\boldsymbol{x}) = -13.401904$) was reported by Ryoo and Sahinidis (1995). Use both the interior-reflective Newton method and the SQP method in *fmincon* to find the global optimum.

Hint: You may need to try different starting points to find the solutions.

58. *Formulation and Solving an ILP Problem*

Three high-value chemical products, *A*, *B*, and *C* are to be produced from two raw materials, namely RawMat 1 and RawMat 2. The production of 1 batch of *A* requires 4 kg of RawMat 1 and 2.5 kg RawMat 2 and 3.5 hours of labor. On the other hand, the production of 1 batch of *B* requires 1 kg of RawMat 1 and 3.5 kg of RawMat 2 and 2 hours of labor. Finally, the production of 1 batch of *C* requires 8.5 kg of RawMat 1 and 4 kg of RawMat 2 and 5 hours of labor.

In terms of resource capacities for one week of production, 1,000 kg of RawMat 1 and 1,200 kg of RawMat 2 are available, while 20 employees each working 40 hours are used.

The following marketing data are available:

	Profit/batch ($)	Demand/week
Product *A*	10.00	40 batches
Product *B*	5.00	100 batches
Product *C*	15.00	30 batches

(a) Formulate this optimization problem in which the goal is to determine the number of batches products *A*, *B*, and *C* are to be produced in a week so as to maximize the total profit. Also, classify the problem. Note that a partial batch cannot be produced.

(b) Solve the ILP problem using LINDO.

59. *Integer Linear Programming*

Solve the following problems via the branch and bound method. You may use LINDO to solve each LP sub-problem (node).

(a) Maximize $f(\mathbf{x}) = 75x_1 + 6x_2 + 3x_3 + 33x_4$

Subject to

$$774x_1 + 76x_2 + 22x_3 + 42x_4 \le 875$$
$$67x_1 + 27x_2 + 794x_3 + 53x_4 \le 875$$
$$x_1, x_2, x_3, x_4 \text{ either 0 or 1}$$

(b) Maximize $f(\mathbf{x}) = 2x_1 + x_2$

Subject to

$$x_1 + x_2 \le 5$$
$$x_1 - x_2 \ge 0$$
$$6x_1 + 2x_2 \le 21$$
$$x_1, x_2 \ge 0 \text{ and integer}$$

60. *Using LINDO to Solve Scheduling Problems*

Consider the following 8-product, 3-unit multistage scheduling problem with unlimited intermediate storage:

	Processing Times (in minutes)							
Product	1	2	3	4	5	6	7	8
Unit 1	20	12	30	45	60	20	15	25
Unit 2	55	30	50	40	30	20	25	32
Unit 3	15	20	38	25	51	14	26	33

Use LINDO to solve for the optimal schedule with the shortest makespan.

61. *Completion Time Calculations*

Consider a 4-stage multiproduct plant with 4 products. The processing time matrix (in minutes) is given below.

	Products			
Units	1	2	3	4
1	5	1	2	2
2	10	3	18	2
3	10	3	10	25
4	32	10	5	15

Assuming the production sequence to be 1-2-3-4 on all four processing units, use Gantt charts to calculate a complete schedule and the makespan for the following cases:

i) UIS policy
ii) NIS policy
iii) ZW policy
iv) MIS (Mixed Intermediate Storage) policy with ZW between units 1 and 2, NIS between units 2-3, and FIS with one storage unit between units 3 and 4.

62. *UIS Scheduling with Makespan Minimization*

Consider the makespan minimization of a 5-product, 3-unit UIS multiproduct plant with unlimited intermediate storage and the following processing time matrix (in minutes):

Units	Products 1	2	3	4	5
1	15	10	10	5	8
2	8	20	15	15	14
3	5	12	16	9	18

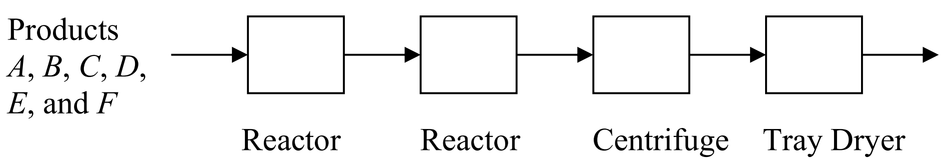
(a) Write a MATLAB code to use brute force (i.e. examine every possible solution) to find a schedule that minimizes the makespan.

(b) Formulate the problem as an MILP, and use LINDO to solve for an optimal schedule in Part (a)

63. *Optimal Scheduling of Products in a Serial Batch Process*

An important optimization problem that arises in a batch chemical plant (or non-continuous processes) is the scheduling problem. Short-term scheduling involves sequencing and scheduling the production of N products across M processing units to optimize a suitable performance or cost-based criterion (Ku *et al.* 1987).

Consider the following serial batch process which consists of 2 batch reactors followed by a centrifuge and a tray dryer. The process is to



manufacture 6 products, namely A, B, C, D, E, and F. All products must follow the same production sequence, i.e. they must first pass through the first reactor, followed by the second, then onto the centrifuge and finally the tray dryer. The processing time of each product required in each processing unit is given in the table below. Each product also has a delivery due date after which a penalty would be incurred.

An effective heuristic (method yielding an approximate and not necessarily optimal solution) called Duedate-Over-Penalty or DOP algorithm [Ku, H.M., and Karimi, I.A., "Scheduling in Serial Multiproduct Batch Processes with Due Date Penalties", **Ind. Eng. Chem. Res.**, **29**, 4, 580, (1990)] has been proposed to solve the due-date scheduling problem as follows:

1. Compute d_i/p_i, $i = 1, N$.

2. Sequence products in the ascending order of d_i/p_i.

3. Compute the completion times and the total penalty cost P for the above sequence.

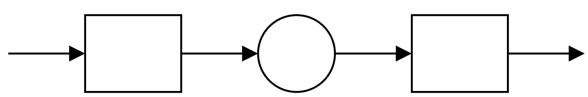
Solve the problem using the DOP algorithm, and compare the solution you obtain with the brute force approach (i.e. examining every possible solution) using MATLAB. Assume that there is unlimited intermediate storage (UIS policy) available to store intermediate products between every 2 stages.

Batch Unit	**Processing Times (Hours)**					
	A	*B*	*C*	*D*	*E*	*F*
Reactor 1	5	35	10	40	25	45
Reactor 2	15	20	30	18	32	7
Centrifuge	10	15	30	12	16	22
Tray Dryer	8	10	20	10	28	38
Due Date, d_i (Days)	1.5	4.0	2.5	5.0	3.0	6.0
Penalty, p_i ($/Hour)	500	400	600	800	300	200

64. *Optimal Scheduling of Products in a Serial Batch Process with FIS Policy*

Now, consider another batch process with only 2 processing stages as shown:

Products
$P1$, $P2$, $P3$,
$P4$, and $P5$



This 2-unit serial system operates under the Finite Intermediate Storage policy (FIS) with one storage vessel, and is used to produce 5 products, namely $P1$, $P2$, $P3$, $P4$, and $P5$. The processing time matrix for this system is given below.

Batch Unit	Processing Times (Minutes)				
	P1	*P2*	*P3*	*P4*	*P5*
Reactor	10	6	2	4	15
Stripper	9	5	14	8	20

For the FIS policy, the following recurrence relations were developed by Ku (1984) to compute the completion time of each product on each processing unit for a given production sequence: k_1 - k_2 - k_3 - … - k_N where product k_i is in the ith position in the production sequence:

$$C_{ij} \;=\; \text{Max}\left[\, C_{(i-1)j} \,,\, C_{i(j-1)} \,,\, C_{(i-1-z\,)(j+1)} - t_{k\,j} \,\right] + t_{k\,j} \qquad i = 1, \ldots, N \quad j = 1, \ldots, M$$

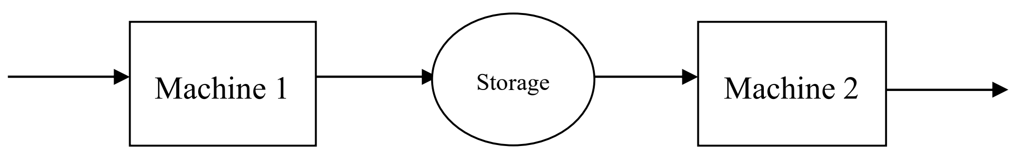
$$C_{ij} \;=\; 0 \;\text{ if }\; i \le 0 \;\text{ or }\; j \le 0$$

where C_{ij} = completion time of the ith product on the jth processing unit
= the time at which the ith product leaves unit j
$t_{k\,j}$ = processing time of product k_i on unit j
z_j = number of storage units available in the FIS policy between unit j and unit (j+1)

Formulate this FIS scheduling problem as an MILP model and determine an optimal production sequence that minimizes the makespan in the production of the 5 products. Also, use MATLAB to implement the brute force approach by examining every possible solution in order to find the same optimal schedule solved in the MILP model.

65. *A Two-Machine Scheduling Problem under UIS and FIS Storage Policies*

Consider a two-machine scheduling problem of 6 products as shown with the following processing time matrix:



	Processing Times t_{ij} Machine 1	2
Product A	50	10
B	10	15
C	15	30
D	26	5
E	6	20
F	8	26

(a) A simple optimization algorithm called Johnson's Rule (March, 1954) exists that gives an optimal sequence with the minimum makespan when there is unlimited intermediate storage (UIS operating policy). The Rule can be stated as follows:

1. Create a list P containing products whose $t_{i1} \leq t_{i2}$ and are arranged in the increasing value of their t_{i1}.
2. Create a second list Q containing products whose $t_{i1} > t_{i2}$ and are arranged in the decreasing value of their t_{i2}.
3. Concatenate P and Q, i.e. combine them so that the optimal sequence is $P \cup Q$.

Use Johnson's Rule to determine the sequence with the minimum makespan in this scheduling problem. Also, calculate this minimum makespan.

(b) Write out the complete formulation for this UIS scheduling problem. In your formulation, you must first define all the variables, and then state the objective function and all the constraints term by term. Do not use any summation sign to show your objective function or constraints. Processing times should also be written down as numbers, not as symbols, in the formulation. Finally, use LINDO to obtain the optimal schedule which should give the same answer as in Part (a).

(c) When the storage operating policy is FIS (finite intermediate storage), the completion time of a product in position i from machine j is given as (Ku and Karimi, 1988):

$$C_{ij} = \max(C_{(i-1)j}, C_{i(j-1)}, C_{(i-z-1)(j+1)} - t_{ij}) + t_{ij}$$

where z = the number of storage vessels, e.g $z = 0$ if there is no storage and $z = \infty$ if there is unlimited intermediate storage. Using the optimal sequence under UIS obtained in Part (a), calculate its makespan when $z = 1$. Note that this sequence is not necessarily an optimal one under the FIS policy.